Isomonodromic deformations and maximally stable bundles

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November 18, 2008

Abstract

We consider irreducible tracefree meromorphic rank 2 connections over compact Riemann surfaces of arbitrary genus. By deforming the curve, the position of the poles and the connection, we construct the universal isomonodromic deformation of such a connection. We prove that the underlying vector bundle is generically maximally stable along the universal isomondromic deformation, provided that the initial connection is irreducible. For surfaces of genus greater than 1, we obtain a non-trivial result even for regular connections.

Contents

1 Introduction

1.1 Result

We consider a meromorphic and tracefree connection ∇_0 on a holomorphic rank 2 vector bundle E_0 over a compact Riemann surface X_0 of genus g. In local trivialization charts for E_0, the connection ∇_0 is defined by $\mathrm{d} - A_0$, where A_0 is a 2×2-matrix whose entries are meromorphic 1-forms such that $\mathrm{tr}(A_0) \equiv 0$. Such a connection (E_0, ∇_0) will be considered up to holomorphic gauge-transformations of the vector bundle. The precise definitions of connections, isomonodromic deformations and related notions we shall use are recalled in section 2.

Roughly speaking, an isomonodromic deformation of $(E_0 \to X_0, \nabla_0)$ is an analytic, topologically trivial deformation $(E_t \to X_t, \nabla_t)_{t\in T}$ such that the monodromy data are constant. By topologically trivial deformation we mean that the associated family $\pi : \mathcal{X} \to T$ of Riemann surfaces with fiber $\pi^{-1}(t) = X_t$ is topologically trivial and is provided with smooth disjoint sections $\mathcal{D}^i : T \to \mathcal{X}$ for $i \in \{1, \ldots, m\}$, which correspond to the polar locus of the family of connections. In the non-singular or logarithmic case (poles of order 1), the monodromy data reduce to the monodromy representation $\pi_1(X_0 \setminus D_0) \to \mathrm{SL}(2, \mathbf{C})$, where D_0 is the polar locus of the initial connection. In this case, a deformation $(E_t \to X_t, \nabla_t)_{t\in T}$ of the curve, the fibre bundle and the connection is called isomonodromic if it is induced by a flat logarithmic connection $(\mathcal{E} \to \mathcal{X}, \nabla)$. In the general meromorphic case (poles of arbitrary order), one usually adds Stokes matrices to the monodromy data (see papers of B. Malgrange, J. Palmer and I. Krichever), therefore needing a non-resonance condition. In the non-resonant case, a deformation $(E_t \to X_t, \nabla_t)_{t\in T}$ is called isomonodromic (and iso-Stokes) if the order of the poles is constant along the deformation, and if it is induced by a flat meromorphic connection over $\mathcal{X}$ whose connection matrix A satisfies

$$(\mathrm{d}A)_\infty \leq (A)_\infty, \tag{1}$$

where $(\cdot)_\infty$ denotes the (effective) polar divisor. If $(x_1, \ldots, x_N)$ are local coordinates in which the polar locus is given by $\{x_1 = 0\}$, then condition (1) means that the connection matrix A takes the form

$$A = M_1 \frac{\mathrm{d}x_1}{x_1^l} + \sum_{i=2}^{N} M_i \frac{\mathrm{d}x_i}{x_1^{l-1}},$$

where M_i, for $i \in \{1, \ldots, N\}$, is a matrix whose entries are holomorphic functions, and l is the order of the pole. If the order of the poles is constant, then it turns out (see section 2.3) that, in the $\mathfrak{sl}(2, \mathbf{C})$-case, condition (1) is equivalent to the existence of local coordinates in which the connection is gauge-equivalent to a constant one:

$$A = M_1(x_1) \frac{\mathrm{d}x_1}{x_1^l}.$$

We shall use this point of view, which is specific to the $\mathfrak{sl}(2,\mathbf{C})$-case, and which enables us to include the $\mathfrak{sl}(2,\mathbf{C})$-resonant case in a natural way.

The first part of this thesis is devoted to the construction of the global universal isomonodromic deformation $(\mathcal{E} \to \mathcal{X}, \nabla)$ over $\mathcal{X} \to T$ of the initial connection. In the non-singular or logarithmic case, the parameter space T is the Teichmüller space $\mathcal{T}$ associated to the punctured curve X_0^* (where punctures are poles of ∇_0) and $(\mathcal{E}, \nabla)$ is the unique flat logarithmic extension of (E_0, ∇_0) over the universal Teichmüller curve of marked punctured Riemann surfaces associated to X_0^*. In the case of multiple poles, there exist non-trivial isomonodromic deformations of the initial connection, which are fixing the curve and the poles. In the general case, the parameter space T of the universal isomonodromic deformation is the product of the Teichmüller space $\mathcal{T}$ of the curve minus the poles, with spaces of convenient jets of diffeomorphisms at the poles. The dimension of this parameter space T is

$$3g - 3 + n, \tag{2}$$

where n is the number of poles counted with multiplicity. This construction, as well as the proof of its universal property will be carried out in section 3.
Such a construction has been done in the non-resonant case for arbitrary rank, using Birkhoff normal form and Stokes matrices, in [Mal83a] (see also [Mal04]),[Mal83b] and [Pal99], for $g = 0$, and in [Kri02], for $g \geq 0$. Our construction does not use Stokes analysis, and is in this sense more elementary, but clearly iso-Stokes in the non-resonant case. In [Mal86] and [Mal96], B. Malgrange gave also a construction of a germ of a universal isomonodromic deformation for resonant singularities, if the leading term of the connection matrix has only one Jordan block for each eigenvalue. In the $\mathfrak{sl}(2,\mathbf{C})$-case, each resonant singularity is clearly of that type. Our elementary approach allows the construction of a global universal isomonodromic deformation even in the resonant case. Another possible approach, that we omit in this work, is the Kodaira-Spencer method. After projectivization of the fibre bundle $\mathcal{E}$, the flat connection ∇ defines a codimension 1 equisingular unfolding on $\mathbf{P}(\mathcal{E})$ in the sense of [Mat91], [MN94]. The obstruction space is given by $\mathrm{H}^1(X_0, \Theta_{X_0}(D_0))$, where D_0 is the effective polar divisor of ∇_0 and Θ is the sheaf of holomorphic vector bundles. The dimension of $\Theta_{X_0}(D_0)$ is $3g - 3 + \deg(D_0)$ (*cf.* [GM89], page 196). The main result of [MN94] insures the existence of a local Kuranishi space. This is a germ of our parameter space.

The second part of this thesis is devoted to the stability of the vector bundle underlying a "generic" irreducible meromorphic rank 2 connection with given monodromy data over a genus g Riemann surface. More precisely, we examine the stability of the underlying vector bundle E_t along the universal isomonodromic deformation $(E_t, \nabla_t)_{t \in T}$ of the initial connection. We define the *degree of stability* $\kappa(E)$ of a rank 2 bundle E on a Riemann surface X as

$$\kappa(E) = \min\{\deg(E) - 2\deg(L) \mid L \text{ sublinebundle of } E\}.$$

When $\kappa(E) > 0$ (resp. $\kappa(E) \geq 0$), the bundle E is called *stable* (resp. *semi-stable*). According to M. Nagata in [Nag70], the degree of stability is upperly bounded by the genus g of X. When $\kappa(E) = g$ or $g - 1$, the bundle E is called *maximally stable*. If E admits a meromorphic connection ∇ with n poles which is *irreducible*, *i.e.* that E has no ∇-invariant sublinebundle L, then we have

$$2 - 2g - n \leq \kappa(E) \leq g,$$

where the first inequality is implied by formula (21), due to M. Brunella (see section 5.2). Note that $\kappa(E)$ is even if E admits a tracefree meromorphic connection. In section 5 we will prove the following main theorem of this thesis.

Theorem 1.1. *Consider the universal isomonodromic deformation*

$$(E_t \to X_t, \nabla_t)_{t \in T}$$

of an irreducible tracefree meromorphic rank 2 connection (E_0, ∇_0) over X_0. Then the vector bundle underlying a generic connection along this deformation is maximally stable. More precisely, for each integer k, the set

$$T_k = \{t \in T \mid \kappa(E_t) \leq k\}$$

is a closed analytic subset of T of codimension at least $g - 1 - k$.

In particular, the vector bundle underlying a (non trivial) isomonodromic deformation of a *non-singular* irreducible $\mathfrak{sl}(2, \mathbf{C})$-connection on a Riemann surface of genus $g \geq 2$ is generically stable.

Corollary 1.2. *Let (E_0, ∇_0) be an irreducible tracefree meromorphic rank 2 connection over $\mathbf{P}^1$. Denote by E_t the vector bundle associated to the parameter $t \in T$ in the universal isomonodromic deformation of (E_0, ∇_0) . Recall that such a vector bundle is of the form $E_t \cong \mathcal{O}(\frac{1}{2}\kappa(E_t)) \oplus \mathcal{O}(-\frac{1}{2}\kappa(E_t))$. Then for a generic parameter $t \in T$, the vector bundle E_t is the trivial one :*

$$E_t = \mathcal{O} \oplus \mathcal{O}.$$

This corollary has been proved for some logarithmic resonant connections by A. Bolibruch in [Bol90][1]. The above result is thus a generalization of a theorem of A. Bolibruch in the spirit of [EV99] and [EH01].

We will give an explicit example of an isomonodromic deformation in section 4, where the exceptional set (called Θ-divisor in [Mal04]) is explicitely given. In section 6, we will weaken the conditions of irreducibility and tracefreeness in theorem 1.1. Namely we will state a sharper version of this result, dealing also with reducible connections, in section 6.1.

1.2 Applications

a) **Isomonodromy equations**
Let (E_0, ∇_0) be an irreducible tracefree rank 2 connection with four poles on $\mathbf{P}^1$, where E_0 is the trivial bundle $\mathbf{P}^1 \times \mathbf{C}^2$. The solutions of the Painlevé equations II-VI describe universal isomonodromic deformations $(E_t, \nabla_t)_{t \in T}$ of such initial systems

[1] On page 37, A. Bolibruch stated that if (E_0, ∇_0) is an irreducible logarithmic rank 2 connection, with at least 4 poles and such that none of its monodromy matrices is diagonalizable and E_0 is a non-maximally stable bundle, then one can choose one pole a_i such that the degree of stablility of the underlying vector bundle can be increased by a small move of a_i in $\mathbf{P}^1$, keeping the monodromy constant.

(E_0, ∇_0). According to corollary 1.2, any universal isomonodromic deformation of a tracefree rank 2 connection with four poles on $\mathbf{P}^1$ contains connections (E_t, ∇_t) which are actually *systems*, *i.e.* whose underlying vector bundles are trivial. In fact this is the case for each generic parameter. In other words, any such universal isomonodromic deformation occurs as a solution of the associated Painlevé equation (see section 4). Of course this result remains true for general Schlesinger equations, which describe isomonodromic deformations of tracefree Garnier systems, see [IKSY91]. More generally, if we want to find explicit isomonodromy equations for genus $g \geq 0$, it is convenient to restrict ourselves to the space of semi-stable vector bundles, permitting to resort to a consistent moduli theory. According to theorem 1.1, this is a natural way of proceeding.

b) **Branched projective structures**

Let X be a compact Riemann surface. The degree of stability of a rank 2 bundle $E \to X$ depends only on the projective bundle $\mathbf{P}(E) \to X$, and it can be identified to the minimal self-intersection number of sections of this latter bundle. Indeed, let L be a sublinebundle of E over X and let σ be the associated section of the projective bundle $\mathbf{P}(E)$. We then have

$$\deg(\det(E)) - 2\deg(L) = \sigma \cdot \sigma, \tag{3}$$

where $\sigma \cdot \sigma$ is the *self-intersection number* of σ (see [Mar70], page 11).

Theorem 1.1 remains valid at the level of projective connections. The proof, given in section 5 will actually be based on projective connections. Recall that the analytic continuations of local solutions of a projective connection (E, ∇) are defining the leaves of a *Riccati foliation* $(P, \mathcal{F})$ on the ruled surface $P = \mathbf{P}(E)$. These foliations are studied thoroughly in [Bru04].

A *projective structure* on a given Riemann surface X of genus g is an atlas of charts in $\mathbf{P}^1$, whose transition maps are Möbius transformations. Equivalently, it can be defined by a Riccati foliation on a $\mathbf{P}^1$-bundle $P \to X$ together with a section $\sigma : X \to P$ transverse to the foliation. To obtain an element of the equivalence class of the associated atlas, an arbitrary fibre of P can be chosen and projection on this fibre along the leaves of the foliation will define the local charts. This construction is also called an $\mathfrak{sl}(2, \mathbf{C})$-oper (see [BD05], page 12).

If one allows tangencies between σ and the foliation, the coordinate maps may become non-conformal. In this way, one gets branch points for the projective structure. The number of branch points is the number of tangencies between the section σ and the foliation. There are $\sigma \cdot \sigma + 2g - 2$ such tangencies (see formula (21)). If κ is the minimal self-intersection number of sections of the bundle P, then each branched projective structure defined by a Riccati foliation on P has at least $\kappa + 2g - 2$ branch points. Each projective structure on X provides a monodromy representation $\rho : \pi_1(X \setminus B) \to \mathrm{PSL}(2, \mathbf{C})$, where B is the branch locus on X. Yet in each branch point, the local monodromy is trivial so that we rather consider the simplified monodromy representation $\rho' : \pi_1(X) \to \mathrm{PSL}(2, \mathbf{C})$, which can be identified with the monodromy representation of the Riccati foliation. Now fix a representation

$\rho' : \pi_1(X) \to \mathrm{PSL}(2, \mathbf{C})$. According to the Riemann-Hilbert correspondence, there is a unique non-singular Riccati foliation $(P \to X, \mathcal{F})$ associated to ρ'. Since the minimal self-intersection number for sections of P is less or equal to g according to M. Nagata, we have the following result.

Proposition 1.3. *There is a branched projective structure on X with simplified monodromy ρ' having at most $3g-2$ branch points, where g is the genus of the compact Riemann surface X.*

As a corollary of theorem 1.1 we now obtain the following theorem.

Corollary 1.4. *If ρ' is irreducible and X is a generic point in the Teichmüller space* $\mathrm{Teich}(g)$*, then each branched projective structure on X with simplified monodromy ρ' has at least $3g-3$ branch points.*

Remark 1.5. *The minimal number of branch points on a generic curve X here is either $3g-2$ or $3g-3$, depending on the parity of the genus g. More exactly, the number of branch points is even, if the simplified monodromy representation ρ' lifts to a representation $\widetilde{\rho'} : \pi_1(X) \to \mathrm{SL}(2, \mathbf{C})$, and odd otherwise.*

2 Definitions and elementary properties

In the following, we shall always denote by M a complex manifold and by X a compact Riemann surface. We denote by $\mathcal{O}$ the sheaf of holomorphic functions on M (resp. X) and by $\Omega \otimes \mathcal{M}$ the sheaf of meromorphic 1-forms on M (resp. X).

2.1 Flat meromorphic connections and monodromy

Let E be a holomorphic rank r vector bundle over M. The bundle E is given by a trivialization atlas (U_i) on M with transition maps φ_{ij}, providing trivialization charts $U_i \times \mathbf{C}^r$ with local coordinates (z_i, Y_i), and transition maps $(\Phi_{ij}) = (\varphi_{ij}, \phi_{ij})$ satisfying

$$(z_i, Y_i) = (\varphi_{ij}(z_j), \phi_{ij}(z_j) \cdot Y_j),$$

where $\phi_{ij} \in \mathrm{GL}(r, \mathcal{O}(U_i \cap U_j))$. Later on, we also denote by E the global space of the vector bundle.

A *meromorphic connection* ∇ on E associates to each trivialization chart $U_i \times \mathbf{C}^r$ of E with coordinates (z_i, Y_i) a system

$$\mathrm{d}Y_i = A_i(z_i) \cdot Y_i \tag{4}$$

with $A_i \in \mathfrak{gl}(2, \Omega \otimes \mathcal{M}(U_i))$, such that the *connection matrices* A_i glue together by means of the transition maps (Φ_{ij}):

$$A_i \circ \varphi_{ij} = \phi_{ij} A_j \phi_{ij}^{-1} + \mathrm{d}\phi_{ij} \phi_{ij}^{-1}.$$

A biholomorphic *coordinate transformation* $\tilde{z}_i = \varphi_i(z_i)$ in the local coordinates of M conjugates the connection matrix $A_i(z_i)$ to

$$\widetilde{A}_i(\tilde{z}_i) = \varphi_i^* A_i(\tilde{z}_i).$$

On the other hand, a holomorphic *gauge transformation* $\widetilde{Y}_i = \phi_i(z_i)Y_i$ with $\phi_i \in \mathrm{GL}(r, \mathcal{O}(U_i))$ on a local chart U_i conjugates system (4) to $\mathrm{d}\widetilde{Y}_i = \widetilde{A}_i(z_i)\widetilde{Y}_i$ with

$$\widetilde{A}_i = \phi_i A_i \phi_i^{-1} + \mathrm{d}\phi_i \phi_i^{-1}.$$

In this article, connections shall be considered modulo holomorphic *gauge-coordinate-transformations*, *i.e.* combination of coordinate and gauge transformations. Two connection over the same base curve M are called *isomorphic*, if, with respect to a common atlas of the base curve, their connection matrices are conjugated by holomorphic gauge transformations.
The poles of the matrices A_i are the *poles* of the connection ∇. They do not depend on the chart, and the *polar divisor* $(\nabla)_\infty$ is well defined. We shall denote by D the *reduced polar divisor*. The connection ∇ is said to be a *non-singular* (resp. *logarithmic*) connection if it has no poles (resp. if it has only simple poles and its connection matrices A_i satisfy $(\mathrm{d}A_i)_\infty \leq (A_i)_\infty$.)

A connection is *flat* or *integrable*, if the connection matrices A_i satisfy $\mathrm{d}A_i \equiv A_i \wedge A_i$. Equivalently, a connection is flat if each non-singular point has a small neighborhood such that there is a gauge transformation $(z, \widetilde{Y}) = (z, \phi(z) \cdot Y)$ which conjugates the connection matrix to the trivial connection matrix $\widetilde{A}_i = 0$.
We can choose a *fundamental solution* S, that is a basis of the space of local solutions in some base point in the set of non-singular points $M^* = M \setminus D$. Then analytic continuation along a closed path γ in the set of non-singular points provides another fundamental solution $S' = \rho^{-1}(\gamma)S$, where $\rho(\gamma)$ is called the *monodromy* along the path γ. In that way we get a *monodromy representation* $\rho : \pi_1(M^*) \to \mathrm{GL}(r, \mathbf{C})$ which will be considered modulo conjugacy of the image of ρ by an element of $\mathrm{GL}(r, \mathbf{C})$.
Each meromorphic rank r connection ∇ on E induces a *trace connection* $\mathrm{tr}(\nabla)$ on the line bundle $\det(E)$, given by

$$\mathrm{d}y_i = \mathrm{tr}(A_i(z_i)) \cdot y_i.$$

We say a connection is *tracefree*, if its trace connection is the trivial connection $\mathrm{d}y = 0$ on the trivial line bundle $M \times \mathbf{C}$. For tracefree connections, it is possible to choose transition maps with $\phi_{ij} \in \mathrm{SL}(r, \mathcal{O}(U_i \cap U_j))$. Thus for tracefree connections we will only consider gauge transformations ϕ_i in $\mathrm{SL}(r, \mathcal{O}(U_i))$.
Let ∇ be a connection on a rank 2 vector bundle E over some Riemann surface X defined by systems (4) and let ζ be a connection on a line bundle L with cocycle λ_{ij} on a common atlas of X, locally defined by

$$\mathrm{d}y_i = a_i(z_i)y_i$$

with $a_i \in \Omega \otimes \mathcal{M}(U_i)$. Then the *tensor product*

$$(L, \zeta) \otimes (E, \nabla)$$

provides a connection on the bundle $L \otimes E$ with cocycle $(\lambda_{ij} \cdot \phi_{ij})$, locally defined by the systems

$$\mathrm{d}\widetilde{Y}_i = (A_i + a_i I)\widetilde{Y}_i.$$

Lemma 2.1. *There is a rank 1 connection (L, ζ) such that $(L, \zeta) \otimes (E, \nabla)$ is tracefree if, and only if, the degree of stability $\kappa(E)$ of E has even parity.*

Proof: If $\kappa(E)$ is even, then the degree of the line bundle $\det(E)$ is even. Thus there is a line bundle L on X such that $L^{\otimes 2} = \det(E)$. This line bundle admits a connection given in trivialization charts by

$$\mathrm{d}y_i = -\frac{1}{2}\mathrm{tr}(A_i)y_i.$$

If $\kappa(E)$ is odd, then for each line bundle L on X the degree of the tensor product $\deg(\det(L\otimes E)) = 2\deg L + \deg(\det(E))$ remains odd. $\square$

2.2 Projective connections and Riccati foliations

Let E be a rank 2 vector bundle on X. To each meromorphic connection (E, ∇) on E we may associate a *projective connection* $(\mathbf{P}(E), \mathbf{P}(\nabla))$ on the $\mathbf{P}^1$-bundle $\mathbf{P}(E)$. Let

$$\mathrm{d}Y = A(x)\cdot Y \quad \text{with} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbf{C}^2$$

be a system defining ∇ on a certain trivialisation chart $U \times \mathbf{C}^2$. Then for the coordinates (x, y) of $U \times \mathbf{P}^1$ with $y = \frac{y_2}{y_1}$ we get

$$\begin{aligned} \mathrm{d}y &= \tfrac{y_1 \mathrm{d}y_2 - y_2 \mathrm{d}y_1}{(y_1)^2} = \tfrac{y_1(cy_1+dy_2)-y_2(ay_1+by_2)}{(y_1)^2} = \\ &= c + (d-a)y - by^2. \end{aligned}$$

The analytic continuations of local solutions of this projective connection are forming the leaves of a so-called Riccati foliation on the corresponding ruled surface (see [Bru04]).
Let P be a $\mathbf{P}^1$-bundle over X. A foliation $\mathcal{F}$ on P is called a *Riccati foliation* when it is defined by meromorphic Riccati equations

$$\mathrm{d}y_i + \alpha_i(x_i)y_i^2 + \beta_i(x_i)y + \gamma_i(x_i) = 0, \quad \alpha_i, \beta_i, \gamma_i \in \Omega \otimes \mathcal{M}(U_i) \tag{5}$$

on charts $U_i \times \mathbf{P}^1$, such that these equations are conjugated by the transition maps $\phi_{ij} \in \mathrm{PGL}(2, \mathcal{O})$ of P. The foliation $\mathcal{F}$ then has vertical leaves located in the poles of α, β, γ. All other fibres of P are globally transverse to $\mathcal{F}$. The singularities of $\mathcal{F}$ are located on the vertical leaves. As before, two Riccati foliations on X are called *isomorphic*, if they are conjugated by holomorphic gauge transformations $(x, Y) \mapsto (x, \phi(x)\cdot Y)$ with $\phi \in \mathrm{PSL}(2, \mathcal{O})$.
On the other hand, let $(\mathcal{F}, P)$ be a Riccati foliation. One can show that for each rank 2 vector bundle E satisfying $\mathbf{P}(E) = P$ and each meromorphic connection ζ on $\det(E)$, there is a unique meromorphic connection ∇ on E such that $\mathbf{P}(\nabla) = \mathcal{F}$ and $\mathrm{tr}(\nabla)) = \zeta$. Two rank 2 connections over a Riemann surface are called *projectively equivalent*, if they define the same Riccati foliation.

Lemma 2.2. *Two rank 2 connections (E, ∇) and $(\widetilde{E}, \widetilde{\nabla})$ over X are projectively equivalent if, and only if, there is a rank 1 connection (L, ζ) over X such that*

$$(\widetilde{E}, \widetilde{\nabla}) = (L, \zeta) \otimes (E, \nabla).$$

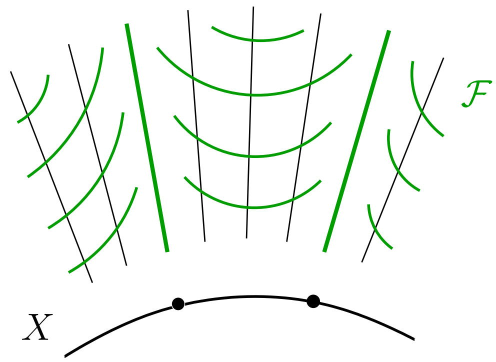

Figure 1: Riccati foliation: A generically transverse foliation

Remark 2.3. *All remains valid on any complex manifold M. The flatness condition for (5) is equivalent to*

$$\begin{array}{rcl} \mathrm{d}\alpha_i & = & \beta_i \wedge \alpha_i \\ \mathrm{d}\beta_i & = & 2\gamma_i \wedge \alpha_i \\ \mathrm{d}\gamma_i & = & \gamma_i \wedge \beta_i \quad . \end{array} \tag{6}$$

2.3 Isomonodromic deformations

Definition 1. *Let $(X_t)_{t\in T}$ be an analytic family of marked Riemann surfaces, given by a submersion $\pi : \mathcal{X} \to T$. Let $(\mathcal{E}, \nabla)$ be a meromorphic connection (not necessarily flat) on $\mathcal{X}$, inducing an analytic family $(E_t \to X_t, \nabla_t)_{t\in T}$. For each parameter $t \in T$, denote by D_t the polar set of the connection $(E_t \to X_t, \nabla_t)$. We only consider the case where $\mathcal{D} = (D_t)_{t\in T}$ is a smooth divisor on $\mathcal{X}$, which is transversal to the parameter t. Then we say that $(E_t \to X_t, \nabla_t)_{t\in T}$ is a* topologically trivial, analytic *family of connections.*

A topologically trivial, analytic family $(E_t \to X_t, \nabla_t)_{t\in T}$ is called an *isomonodromic family*, if it is induced by a flat connection over the total space $\mathcal{X} \to T$ of the family of curves. Along an isomonodromic family, the monodromy representation is constant. An isomonodromic deformation is a special case of isomonodromic families, which is induced by some initial connection $(E_0 \to X_0, \nabla_0)$, such that the Stokes-data are also constant along the deformation. Yet Stokes data are well-defined only in the *non-resonant* case, *i.e.* if the leading term of the connection matrix of the initial connection has only distinct eigenvalues.

Usually (*c.f.* [Mal83a], [Mal83b], [Pal99], [Kri02])), isomonodromic deformations of tracefree rank 2 connections are defined as follows.

Definition 2. *Let $(E_0 \to X_0, \nabla_0)$ be a non-resonant, tracefree rank 2 connection on a Riemann surface X_0. A topologically trivial, analytic deformation $(E_t \to X_t, \nabla_t)_{t\in T}$ of this initial connection is called an* isomonodromic deformation*, if*

- *for each parameter t, the order of the poles of ∇_t is equal to the order of the poles of ∇_0 and*

- *$(E_t \to X_t, \nabla_t)_{t\in T}$ is induced by a flat connection $(\mathcal{E} \to \mathcal{X}, \nabla)$,*

- *whose connection matrix A satisfies the following transversality-condition :*

$$(\mathrm{d}A)_\infty \leq (A)_\infty. \tag{7}$$

Remark 2.4. *Consider a flat tracefree connection of rank 2 on a smooth family of vector bundles over Riemann surfaces with smooth polar divisor $\mathcal{D}$ (as a set). Then any irreducible component $\mathcal{D}^i$ of the polar divisor $\mathcal{D}$ not satisfying the transversality condition (7) is projectively apparent in the following sense : after a bimeromorphic transformation, the polar divisor of the associated projective connection becomes $\mathcal{D} \setminus \mathcal{D}^i$ (as a set) (see [LP07], page 736).*

Let $(\mathcal{E} \to \mathcal{X}, \nabla)$ be a flat connection inducing an isomonodromic family $(E_t \to X_t, \nabla_t)_{t\in T}$. Locally in each non-singular point of $\mathcal{X}$, the connection is given by systems $\mathrm{d}Y \equiv 0$. In particular, any local solution in a non-singular point is automatically transverse to the parameter $t \in T$, *i.e.* transverse to $\{t = \mathrm{const}\}$. Recall that we are considering smooth families $\mathcal{X} \setminus \mathcal{D}$ of marked punctured Riemann surfaces. This implies, that the reduced divisor $\mathcal{D}$ of ∇ is transverse to the parameter. If $\mathcal{D}^i$ is a logarithmic singularity of ∇, then the connection is «locally constant» along $\mathcal{D}^i$ (see section 3.1b)). We shall see that if $(\mathcal{E} \to \mathcal{X}, \nabla)$ defines an isomonodromic deformation, then it is «locally constant» in any point of its polar divisor $\mathcal{D}$. Let us now precise the notion of local constancy. On smooth families $(X_t)_{t\in T}$ of marked Riemann surfaces we shall always denote by $(t, x) \in W \times U$ local trivialization coordinates, with $t_1 \in W \subset T$ and $x \in U \subset X_{t_1}$.

Convention 2.5. *By gauge-coordinate-transformations in coordinates $(t, x, Y) \in W \times U \times \mathbf{C}^2$ with $W \subset T$, we will always mean gauge-coordinate-transformations fixing the parameter t:*

$$(\tilde{t}, \tilde{x}, \widetilde{Y}) = (t, \varphi(t, x), \phi(t, x) \cdot Y).$$

Definition 3. *A flat connection ∇ on a smooth family of holomorphic vector bundles $(E_t \to X_t)_{t\in T}$ is called* locally constant *if locally in each point of the total space $\mathcal{X}$ of the curve deformation, the connection matrix does not depend on the parameter $t \in T$, up to a convenient gauge-coordinate transformation.*

Remark 2.6. *In other words, on open sets as above, there are submersions $\varphi : W \times U \to U$ transversal to the parameter, such that ∇ is gauge-equivalent to the pull-back $\varphi^*(\nabla|_{t=t_1})$.*

This means that up to an appropriate gauge-coordinate transformation on $W \times U \times \mathbf{C}^2$, the system $\mathrm{d}Y = A(x)Y\mathrm{d}x$ defining $\nabla|_{t=t_1}$ over U defines ∇ over $W \times U$ as well. In other words, if ∇ can locally be seen as the product of an initial connection with the parameter space.

Proposition 2.7. *Let $(\mathcal{E} \to \mathcal{X}, \nabla)$ is a flat tracefree rank 2 connection satisfying condition (7) and the leading term of the matrix A has no zeros along the polar divisor, then the connection ∇ is locally constant.*

On the other hand, if ∇ is a flat, locally constant connection on $\mathcal{E} \to \mathcal{X}$, then in every chart U the connection matrix A of ∇ satisfies the transversality condition (7).

Remark 2.8. *Keep in mind that the transversality condition is strictly weaker to the condition of local constancy, if we consider connections of rank greater than 2 or rank 2 connections with non-trivial trace.*

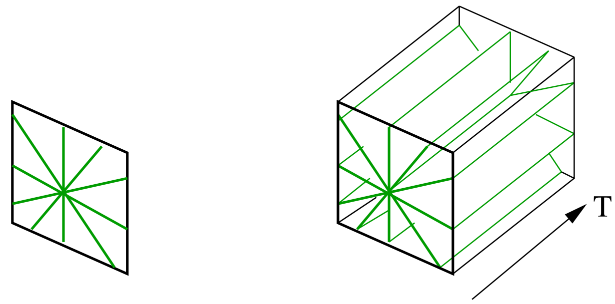

Figure 2: Local constancy, or local product structure

In order to prove proposition 2.7, we will use the following lemma.

Lemma 2.9. *If $(E_t \to X_t, \nabla_t)_{t\in T}$ is a flat tracefree rank 2 connection satisfying the transversality condition (7) such that the order of the poles is constant, and if t_1 is a point of T, then there is a neighborhood W of t_1 such that, in restriction to the parameter space W, the connection ∇ can locally be defined on open sets $W \times U \times \mathbf{C}^2$ with coordinates (t, x, Y) by systems of normal form*

$$\mathrm{d}Y = \frac{1}{x^l}\begin{pmatrix} 0 & 1 \\ c(x) & 0 \end{pmatrix} Y \mathrm{d}x \tag{8}$$

not depending on t, where c is a holomorphic function on U.

Proof: On a local chart $W \times U \times \mathbf{C}^2$, let ∇ be given by a system

$$\mathrm{d}Y = A(t,x)Y \quad \text{with} \quad A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

If ∇_t has a pole of order l at $\{x = 0\}$ for each parameter t in a neighborhood of t_1, then at least one of the 1-forms a, b or c has a pole of order l at $\{x = 0\}$, which remains a pole of order l in restriction to t_1. We may suppose this is the case for b, otherwise we may apply a gauge transformation such as $\widetilde{Y} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) Y$ or $\widetilde{Y} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) Y$. If the transversality condition is satisfied, then b is of the form $b = \frac{1}{x^l}(b_0(t,x)\mathrm{d}x + x b_1(t,x)\mathrm{d}t)$, where b_0 and b_1 are holomorphic functions with $b_0(t,0) \not\equiv 0$. By our assumption, $b_0(t,0)$ is even non-zero for each parameter t in a small neighborhood of t_1. Thus $x^l b$ is defining a non-singular (integrable) foliation transverse to the parameter t, and there is a coordinate transformation fixing $\{x = 0\}$ and straightening the reduced version of $x^l b = 0$ to $\mathrm{d}x = 0$.

Remark 2.10. *With the notions of remark 2.6, the submersion $\varphi(t,x)$ defining this coordinate change is given by a first integral of the foliation $x^l b = 0$.*

In other words, up to a coordinate transformation, we make sure that b is of the form $b = \frac{1}{x^l} b_0(t,x)\mathrm{d}x$, where $b_0(t,x)$ has no zéros in a sufficiently small neighborhood of $(0, t_1)$. Modulo a gauge transformation of the form

$\widetilde{Y} = \begin{pmatrix} (\sqrt{b_0})^{-1} & 0 \\ 0 & \sqrt{b_0} \end{pmatrix} Y$ we may suppose $b = \frac{1}{x^l}\mathrm{d}x$. The integrability condition $\mathrm{d}A = A \wedge A$ is equivalent to

$$\begin{array}{rcl} \mathrm{d}a & = & b \wedge c \\ \mathrm{d}b & = & 2a \wedge b \\ \mathrm{d}c & = & 2c \wedge a \quad . \end{array} \tag{9}$$

We conclude that a has the form $\frac{a_0(t,x)}{x^l}\mathrm{d}x$. By a gauge transformation $\widetilde{Y} = \left(\begin{smallmatrix} 1 & 0 \\ a_0 & 1 \end{smallmatrix}\right) Y$ we make sure that $a \equiv 0$. Then by (9) we have $0 = b \wedge c$. Thus c has the form $\frac{c_0(t,x)}{x^l}\mathrm{d}x$. Again by (9) we get $\mathrm{d}c = 0$. Therefore c does not either depend on t: we have $c = \frac{c_0(x)}{x^l}$. □

Proof of proposition 2.7: Note first that (7) is satisfied if, and only if, it is satisfied after a gauge-coordinate transformation.
Clearly, systems of normal form (8) are locally constant.
Conversely, if ∇ is locally constant, the local charts can be chosen in a way that the connection matrices A do not depend on the parameter t :

$$\mathrm{d}Y = A(x)Y.$$

Then $\mathrm{d}A$ is zero and has no polar divisor. □

Remark 2.11. *Let $(E \to X, \nabla)$ be a meromorphic, tracefree rank 2 connection over a Riemann surface. The upper proof shows that, up to an appropriate holomorphic gauge-transformation, this connection is given locally by systems of normal form (8). Then the rational number* $\max\{l + 1 - \frac{\nu}{2}, 0\}$*, where ν is the greatest integer such that $\frac{c}{x^\nu}$ is still holomorphic, is called the* Katz-rank *of the singularity (*cf. *[Var96]). The Katz-rank is invariant under meromorphic gauge-transformation. Moreover, we see that the Katz-rank is constant along isomonodromic deformations.*

According to proposition 2.7, isomonodromic deformations of tracefree rank 2 connections may be defined alternatively as follows :

Definition 4. *A topologically trivial, analytic deformation $(E_t \to X_t, \nabla_t)_{t \in T}$ of some initial tracefree rank 2 connection $(E_0 \to X_0, \nabla_0)$ is called an* isomonodromic deformation*, if it is induced by a flat, locally constant connection $(\mathcal{E} \to \mathcal{X}, \nabla)$.*

In this thesis, we shall use this latter definition of isomonodromic deformation, which is also valid in the resonant case.

Two isomonodromic deformations of a commun initial connection will be called *isomorphic*, if the associated flat connections are isomorphic. More explicitely, two isomonodromic deformations $(\mathcal{E} \to \mathcal{X}, \nabla)$ and $(\widetilde{\mathcal{E}} \to \widetilde{\mathcal{X}}, \widetilde{\nabla})$ with parameter spaces T, respectively $\widetilde{T}$ are isomorphic, if there is a biholomorphism f, an isomorphism F of marked curves and an isomorphism Ψ of vector bundles extending the isomorphism of the initial connections, such that the following diagramm commutes

$$\begin{tikzcd}
(\mathcal{E},\nabla) \arrow[r, "\Psi", "\sim"'] \arrow[d] & (\widetilde{\mathcal{E}},\widetilde{\nabla}) \arrow[d] \\
\mathcal{X} \arrow[r, "F", "\sim"'] \arrow[d] & \widetilde{\mathcal{X}} \arrow[d] \\
T \arrow[r, "f", "\sim"'] & \widetilde{T}.
\end{tikzcd}$$

Here Ψ consists locally of gauge-coordinate-transformations compatible with F which are conjugating ∇ to $\widetilde{\nabla}$.

Remark 2.12. *We would like to stress that the gauge transformations we defined for these flat families of connections (resp. Riccati foliations) are not only holomorphic families of gauge transformations, but holomorphic gauge transformations of the flat connection on the global bundle $\mathcal{E}$ (resp. $\mathcal{P}$) of the family.*

2.4 Bimeromorphic and elementary transformations

Let us introduce the notion of elementary transformations along a rank 2 vector bundle $\mathcal{E} \to \mathcal{X}$ on a smooth family of Riemann surfaces $\mathcal{X} \to T$ with parameter space T. Let $\mathcal{D}$ be a smooth irreducible divisor on $\mathcal{X}$ transverse to the parameter $t \in T$ and let $s : \mathcal{D} \to \mathcal{E}$ be a holomorphic section without zeros.

Definition 5. *Let $\mathcal{E} \to \mathcal{X}$ and $s : \mathcal{D} \to \mathcal{E}$ be as above. In a neighborhood of the divisor $\mathcal{D}$, choose a trivialization chart $W \times U \times \mathbf{C}^2$ of the vector bundle $\mathcal{E}$ with coordinates $(t, x, (y_1, y_2))$, such that the divisor $\mathcal{D}$ and the section s are given respectively by $\{x = 0\}$ and $\{x = 0, (y_1, y_2) = (1, 0)\}$. In particular, $\mathcal{D}$ and s do not depend on the parameter in these coordinates. Then the* elementary transformation of $\mathcal{E}$ directed by s *is a bimeromorphic map* $\mathrm{elm}_s : \mathcal{E} \dashrightarrow \widehat{\mathcal{E}}$ *, given in the upper coordinates by the bimeromorphic gauge transformation*

$$(t, x, (y_1, y_2)) \mapsto (\hat{t}, \hat{x}, (\hat{y}_1, \hat{y}_2)) = (t, x, (xy_1, y_2)),$$

and by the identity map in every chart over $\mathcal{X} \setminus \mathcal{D}$.

The vector bundle $\widehat{\mathcal{E}}$ is thereby well defined modulo isomorphism : as can be seen in the following commuting diagram, modulo holomorphic gauge transformations the elementary transformation elm_s does not depend on the chosen trivialization chart.

$$\begin{tikzcd}
(x, Y) \arrow[r, "\mathrm{elm}"] \arrow[d, "\left(\varphi(t,x), \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) Y\right)"'] & \left(x, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} Y\right) \arrow[d, "\left(\varphi(x,t), \left(\begin{smallmatrix} 1 & \varphi b \\ 0 & d \end{smallmatrix}\right) Y\right)"] \\
\left(\varphi(t,x), \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} Y\right) \arrow[r, "\mathrm{elm}"] & \left(\varphi(t,x), \begin{pmatrix} \varphi(t,x) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} Y\right)
\end{tikzcd} \tag{10}$$

Here $\varphi(t, x)$ is a diffeomorphism fixing $\{x = 0\}$ and b, d are holomorphic functions on $W \times U$, such that d has no zeros. Note that in the above coordinates, the elementary

transformation centered in $\{x = 0, (y_1, y_2) = (0, 1)\}$ is "inverse" to elm_s in the sense that their composition provides a bundle projectively equivalent to the initial one.
In restriction to a fixed parameter $t_1 \in T$, this definition of elementary transformations on $(E_t \to X_t)_{t \in T}$ is equivalent to the construction of O. Gabber, explained in [EV99] and [Mac07]. Let $\widehat{E}_{t_1} \to X_{t_1}$ be the vector bundle resulting from such an elementary Gabber transformation on $E_{t_1} \to X_{t_1}$, centered in some point p of E_{t_1}. We then have

$$\deg(\det \widehat{E}_{t_1}) = \deg(\det E_{t_1}) + 1$$

and for each line bundle $\widehat{L}$ of $\widehat{E}_{t_1}$ coming from a line bundle L of E_{t_1}, we have

$$\begin{array}{ll} \deg(\widehat{L}) = \deg(L) + 1 & \text{if } p \in L \\ \deg(\widehat{L}) = \deg(L) & \text{if } p \notin L. \end{array}$$

In particular, if $\widehat{\sigma}$ (resp. σ) are the sections associated to $\widehat{L}$ (resp. L) on the respective projective bundles, by (3) their self-intersection numbers are related in the following way:

$$\widehat{\sigma} \cdot \widehat{\sigma} = \begin{cases} \sigma \cdot \sigma - 1 & \text{if } p \in \sigma \\ \sigma \cdot \sigma + 1 & \text{if } p \notin \sigma. \end{cases} \tag{11}$$

This definition is compatible with the usual definition of elemetary transformations on the associated ruled surfaces (see [Fri98], [EV99]).

Remark 2.13. *Each bimeromorphic gauge transformation on a curve is projectively equivalent to the composition of a finite number of elementary transformations (see [LP07], page 737).*

Diagram (10) further shows that elementary transformations elm_s are well defined for vector bundles equipped with a connection $(\mathcal{E} \to \mathcal{X}, \widehat{\nabla})$ and they provide a new connection $(\widehat{\mathcal{E}} \to \mathcal{X}, \widehat{\nabla})$. With the help of a similar commuting diagram, it can be shown that this connection only depends on the position of s in the associated Riccati foliation.

Lemma 2.14. *Denote by $(\widehat{\mathcal{E}}, \widehat{\nabla})$ the flat connection resulting from an elementary transformation along $s : \mathcal{D} \to \mathcal{E}$ on some initial flat connection $(\mathcal{E}, \nabla)$, where $\mathcal{D}$ is a divisor transverse to the parameter.*

1. *Suppose that ∇ is non-singular on $\mathcal{D}$. Then $\widehat{\nabla}$ is locally constant if, and only if, the induced section s of $\mathbf{P}(\mathcal{E})$ is included in a leaf of the associated Riccati foliation.*

2. *Suppose that ∇ is locally constant and has a pole on $\mathcal{D}$. If s is a singularity of the foliation $\mathbf{P}(\nabla)$, then $\widehat{\nabla}$ will still be locally constant.*

Proof: Choose appropriate local coordinates, such that the connection matrix of ∇ and the position of $\mathcal{D}$ are constant in t. Then s is included in a leaf of the associated Riccati foliation if, and only if, s is constant in t with respect to the projective coordinates. Moreover, if s is a singularity of the foliation $\mathbf{P}(\nabla)$, then s is also constant in t with respect to the projective coordinates. Recall that an elementary transformation only depends on the position of its center in the associated Riccati foliation. In both cases, the elementary transformation elm_s thus can be seen as an elementary transformation without parameter.

It is clear that the resulting connection $\widehat{\nabla}$ will then be locally constant. Conversely, if $\widehat{\nabla}$ is constant in some appropriate coordinates, and the connection resulting from the inverse elementary transformation elm_s^{-1} is non-singular over $\mathcal{D}$ as in 1., then the center of elm_s^{-1} has to be a singularity of $\mathbf{P}(\widehat{\nabla})$ over $\mathcal{D}$. Thus the position of the center of elm_s^{-1} is constant, too. By consequence, ∇ and the position of s in $\mathbf{P}(\mathcal{E})$ are constant in the corresponding coordinates. □

In example 5.8 we will construct a non-trivial isomonodromic deformation by an elementary transformation over a pole which is not centered in the singularity.

3 The universal isomonodromic deformation

Let X_0 be a Riemann surface of genus g. Let ∇_0 be a meromorphic tracefree rank 2 connection on $E_0 \to X_0$ with m poles of multiplicity respectively $n_1, \ldots n_m$, given in local coordinates x^i by $\{x^i = 0\}$. Denote by $n = n_1 + \ldots + n_m$ the number of poles counted with multiplicity. We shall denote by $D_0 = \sum_{i=1}^n \{x^i = 0\}$ the polar set on X_0 of this connection. By X_0^* we denote the set of non-singular points $X_0 \setminus D_0$. Let $\rho : \pi_1(X_0^*) \to \mathrm{SL}(2, \mathbf{C})$ be the monodromy representation of ∇_0.

We will now construct the universal isomonodromic deformation $(\mathcal{E} \to \mathcal{X}, \nabla)$ with base curve $\mathcal{X} \to T$ of $(E_0 \to X_0, \nabla_0)$. We will denote by $\mathcal{D} = (D_t)_{t \in T}$ the polar locus of the universal isomonodromic deformation, given by disjoint sections $\mathcal{D}^i : T \to \mathcal{X}$ with $i \in \{1, \ldots, m\}$. The parameter space T will be simply connected, of dimension

$$\dim(T) = \sup(0, 3g - 3 + n),$$

except for the special case $g = 1, n = 0$, where we will have $\dim(T) = 1$.

Convention 3.1. *Since in isomonodromic deformations we associate one monodromy representation to a family of connections, we always will consider the m-punctured base curve $(X_t^*)_{t \in T}$ with $X_t^* = X_t \setminus D_t$ as a marked curve, even when it is not explicitly mentioned.*

The universal isomonodromic deformation will satisfy the following universal property.

Proposition 3.2 (Universal property). *Let $(\widetilde{\mathcal{E}} \to \widetilde{\mathcal{X}}, \widetilde{\nabla})$ be an isomonodromic deformation of $(\widetilde{E}_0 \to \widetilde{X}_0, \widetilde{\nabla}_0)$ with simply connected parameter space $\widetilde{T}$ and initial parameter $\tilde{t}_0$. Suppose there is an isomorphism $F_0 : (\widetilde{X}_0, \widetilde{D}_0) \xrightarrow{\sim} (X_0, D_0)$ of marked curves and an isomorphism $\Psi_0 : (\widetilde{E}_0, \widetilde{\nabla}_0) \xrightarrow{\sim} (E_0, \nabla_0)$ of connections given locally by holomorphic gauge transformations ψ_0 conjugating $(\widetilde{E}_0, \widetilde{\nabla}_0)$ to $F_0^*(E_0, \nabla_0)$.*

$$\begin{tikzcd}
(\widetilde{E}_0, \widetilde{\nabla}_0) \arrow[r, "\Psi_0", "\sim"'] \arrow[d] & (E_0, \nabla_0) \arrow[d] \\
(\widetilde{X}_0, \widetilde{D}_0) \arrow[r, "F_0", "\sim"'] \arrow[d] & (X_0, D_0) \arrow[d] \\
\{\tilde{t}_0\} \arrow[r, "f_0"] & \{t_0\}
\end{tikzcd}$$

Then there is a triple (f, F, Ψ) extending (f_0, F_0, Ψ_0) to a commuting diagramm

$$\begin{tikzcd}
(\widetilde{\mathcal{E}}, \widetilde{\nabla}) \arrow[r, "\Psi"] \arrow[d] & (\mathcal{E}, \nabla) \arrow[d] \\
(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \arrow[r, "F"] \arrow[d] & (\mathcal{X}, \mathcal{D}) \arrow[d] \\
\widetilde{T} \arrow[r, "f"] & T,
\end{tikzcd}$$

where (f, F) are holomorphic maps such that $F|_{\tilde{t}}$ is a biholomorphism of marked Riemann surfaces for each parameter $\tilde{t} \in \widetilde{T}$, and Ψ is given locally by holomorphic gauge transformations ψ conjugating $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ to $F^*(\mathcal{E}, \nabla)$. Moreover, the triple (f, F, Ψ) is unique provided that (g, m) is different from $(0,0), (0,1), (0,2)$ and $(1,0)$.

In the non-singular or logarithmic case, the maps (f, F) will be given by Teichmüller theory. In the general case, the maps (f, F) factorize by the Teichmüller classifying maps. These are not unique in the special cases $(g, m) = (0,0), (0,1), (0,2)$ or $(1,0)$. Yet for each appropriate choice of these maps, the triple (f, F, Ψ) is unique.

This universal property has two immediate corollaries.

Corollary 3.3. *Suppose that (g, m) is different from $(0,0), (0,1), (0,2)$ and $(1,0)$. If T' is a germ of submanifold of T such that the restriction of the universal isomonodromic deformation $(E_t \to X_t, \nabla_t)_{t \in T}$ to T' is* trivial*, i.e. isomorphic to a constant deformation, then*

$$\dim(T') = 0.$$

Proof: Let $(\widetilde{\mathcal{E}} \to \widetilde{\mathcal{X}}, \widetilde{\nabla})$ the the restriction of the universal isomonodromic deformation to the parameter space T'. Then there are two triples (f, F, Ψ) possible in the universal property theorem : f can either be the inclusion map or the constant map. Since we are in the general cases where (f, F, Ψ) is unique, this implies $\dim(T') = 0$. □

Corollary 3.4. *The universal isomonodromic deformation $(\mathcal{E} \to \mathcal{X}, \nabla)$ of $(E_0 \to X_0, \nabla_0)$ is also the universal isomonodromic deformation of each of the connections $(E_t \to X_t, \nabla_t)$ it contains for a parameter $t \in T$.*

Proof: Choose any connection $(E_{t_1} \to X_{t_1}, \nabla_{t_1})$ associated to a parameter $t_1 \in T$. Let $(\widetilde{\mathcal{E}} \to \widetilde{\mathcal{X}}, \widetilde{\nabla})$ be an isomonodromic deformation, such that the connection associated to the initial parameter is isomonorphic to $(E_{t_1} \to X_{t_1}, \nabla_{t_1})$. From the of the universal property theorem, it will follow immediately that this isomorphism extends in a similar to the upper theorem. □

3.1 Construction

Consider the universal curve of marked m-pointed Riemann surfaces $\mathcal{X}_{\mathcal{T}} \to \mathcal{T}$, parametrized by $(\mathcal{T}, \tau_0)$ being the Teichmüller space $\mathrm{Teich}(g,m)$ with initial parameter τ_0 corresponding to X_0 with the distinguished set D_0, as in [Nag88], page 322.

Remark 3.5. *The dimension of the Teichmüller space* $\mathrm{Teich}(g,m)$ *is*

$$\begin{array}{cl} 3g-3+m & \textit{if } g \geq 2 \\ \sup\{m,1\} & \textit{if } g=1 \\ \sup\{m-3,0\} & \textit{if } g=0 \end{array}$$

We shall denote by $\mathcal{D}_{\mathcal{T}} = \sum_{i=1}^{m} \mathcal{D}_{\mathcal{T}}^i$ the submanifold of $\mathcal{X}_{\mathcal{T}}$ corresponding to the distinguished points and their deformations. For $(\mathcal{X}_{\mathcal{T}}, \mathcal{D}_{\mathcal{T}}) = (X_\tau, D_\tau)_{\tau \in \mathcal{T}}$ we will denote by $\mathcal{X}_{\mathcal{T}}^*$ (resp. X_τ^*) the punctured curves $\mathcal{X}_{\mathcal{T}} \setminus \mathcal{D}_{\mathcal{T}}$ (resp. $X_\tau \setminus D_\tau$). Consider the exact sequence of homotopy groups associated to the fibration $(\mathcal{X}_{\mathcal{T}} \setminus \mathcal{D}_{\mathcal{T}}) \to \mathcal{T}$. Since the Teichmüller space $\mathcal{T}$ is contractile (*cf.* [Hub06], page 274), we see that for each parameter $\tau \in \mathcal{T}$, the complex manifolds X_τ^* and $\mathcal{X}_{\mathcal{T}}^*$ are homotopically equivalent and each generator of the fundamental group of $\mathcal{X}_{\mathcal{T}}^*$ corresponds to a unique generator of the fundamental group of X_τ^*. In that way, we can consider ρ as a representation of $\pi_1(\mathcal{X}_{\mathcal{T}}^*)$ or $\pi_1(X_\tau^*)$ as well.

As we shall see, a logarithmic connection (E_0, ∇_0) on X_0 with poles in D_0 extends in a unique way to an integrable logarithmic connection on $\mathcal{X}_{\mathcal{T}}$ with poles in $\mathcal{D}_{\mathcal{T}}$ and induces for each parameter $\tau \in \mathcal{T}$ a unique connection on X_τ with simple poles in D_τ. This defines the universal isomonodromic deformation in the logarithmic case.
In the case of a non-logarithmic initial connection, there is still an integrable locally constant connection on $\mathcal{X}_{\mathcal{T}}$ with poles in $\mathcal{D}_{\mathcal{T}}$ extending $(E_0 \to X_0, \nabla_0)$, but this connection will no longer be unique. Indeed, each pole of order l will contribute $l-1$ degrees of freedom in the construction. We thereby get a universal isomonodromic deformation of dimension

$$3g-3+m+(n-m) = 3g-3+n$$

if $(g,m) \neq (0,0),(0,1),(0,2),(1,0)$. The universal isomonodromic deformation will be global, due to the existence of *tubular neighborhoods* $\mathcal{U}_{\mathcal{T}}^i$ of $\mathcal{D}_{\mathcal{T}}^i$ in $\mathcal{X}_{\mathcal{T}}$.

a) **Regular case (arbitrary rank)**

Theorem 3.6 (Classical Riemann-Hilbert correspondence)**.** *The set of non-singular integrable rank r connections on a complex manifold M (modulo holomorphic gauge transformation) is in one-to-one correspondence with the set of representations from $\pi_1(M, z_0)$ to* $\mathrm{GL}(r, \mathbf{C})$ *(modulo conjugacy).*

Proof: For a given monodromy ρ, we may construct the associated connection over M by *suspension*. Let $\widetilde{M}$ be the universal cover of the complex manifold M. Let $\widetilde{\nabla}$ be the trivial connection $\mathrm{d}\widetilde{Y} = 0$ on the trivial bundle $\widetilde{M} \times \mathbf{C}^r$ with coordinates $(\tilde{z}, \widetilde{Y})$ over $\widetilde{M}$. Now the fundamental group $\pi_1(M)$ is naturally acting on $\widetilde{M}$. We may further define an action on $\widetilde{M} \times \mathbf{C}^r$ in the following way :

$$\gamma \cdot (\tilde{x}, \widetilde{Y}) = (\gamma \cdot \tilde{z}, \rho(\gamma) \cdot \widetilde{Y}).$$

Since the monodromy matrices $\rho(\gamma)$ are constant, the connection $\mathrm{d}\widetilde{Y} = 0$ can be naturally pushed down to the quotient of this action. Thereby we define a non-singular connection on a implicitely defined vector bundle E over M. This connection has monodromy ρ.

Now let (E, ∇) and $(\widetilde{E}, \widetilde{\nabla})$ be two non-singular integrable connections of rank r over M with monodromy representation ρ. We may choose a common atlas for M. Let U be a small neighborhood of $z_0 \in M$. Then up to gauge transformations $\phi(z) \cdot Y$ respectively $\widetilde{\phi}(z) \cdot \widetilde{Y}$, the connections ∇ and $\widetilde{\nabla}$ are defined in the trivialization charts by $\mathrm{d}Y = 0$ respectively $\mathrm{d}\widetilde{Y} = 0$ over U. Now the gauge transformation $\psi = \widetilde{\phi} \circ \phi^{-1}$ can be continued analytically and since the analytic continuations of ϕ and $\widetilde{\phi}$ give rise to the same monodromy representation, ψ has trivial monodromy. Thus ψ defines an isomorphism. $\square$

Corollary 3.7. *Let $Z \subset M$ be a complex submanifold of M such that the inclusion map $i : Z \to M$ provides an isomorphism $i_* : \pi_1(Z, z_0) \xrightarrow{\sim} \pi_1(M, z_0)$ with $z_0 \in Z$. Let (E_Z, ∇_Z) be a flat non-singular rank r connection over Z. Then this connection extends to a flat non-singular rank r connection (E, ∇) over M, which is unique in the following sense.*
If (E, ∇) and $(\widetilde{E}, \widetilde{\nabla})$ are two non-singular rank r connections over M such that in restriction to Z there is an isomorphism $\psi_Z : (\widetilde{E}, \widetilde{\nabla})|_Z \xrightarrow{\sim} (E, \nabla)|_Z$, then ψ_Z extends to an isomorphism $\psi : (\widetilde{E}, \widetilde{\nabla}) \xrightarrow{\sim} (E, \nabla)$ over M.

Proof: Let ρ be the monodromy representation of the connection (E_Z, ∇_Z). Then the classical Riemann-Hilbert correspondence provides a flat non-singular connection (E, ∇) over M, having monodromy ρ. Since (E_Z, ∇_Z) and $(E, \nabla)|_Z$ are both flat non-singular connections on Z having the same monodromy, they are isomorphic.
Let (E, ∇) and $(\widetilde{E}, \widetilde{\nabla})$ be as above. Let $\mathcal{U}$ be a small neighborhood of z_0 in M and let U_Z be the induced neighborhood on Z. In appropriate coordinates, the connections $\widetilde{\nabla}$ and ∇ are given by the trivial connection $\mathrm{d}\widetilde{Y} = 0$ respectively $\mathrm{d}Y = 0$ on $\mathcal{U}$. Denote by ϕ_Z be the gauge transformation corresponding to the restriction of the isomorphism $\psi_Z|_{U_Z}$. Now each gauge transformation conjugating $\mathrm{d}\widetilde{Y} = 0$ to $\mathrm{d}Y = 0$ is constant. Hence there is a unique gauge transformation ϕ over $\mathcal{U}$ conjugating $\widetilde{\nabla}$ to ∇, such that $\phi|_{U_Z} = \phi_Z$. Now ϕ can be continued analytically along any path in M. Since any homotopy class of a closed path in M has a representant in $\pi_1(Z)$, the analytic continuations of ϕ cannot have monodromy. This defines the desired isomorphism ψ. $\square$

If ∇_0 is non-singular, this corollary provides the unique non-singular flat rank 2 connection $(\mathcal{E} \to \mathcal{X}_{\mathcal{T}}, \nabla)$ over the universal Teichmüller curve $\mathcal{X}_{\mathcal{T}} \to \mathcal{T}$, which has the same monodromy representation as $(E_0 \to X_0, \nabla_0)$. We have

$$(\mathcal{E} \to \mathcal{X}_{\mathcal{T}}, \nabla)|_{\tau = \tau_0} = (E_0 \to X_0, \nabla_0).$$

Then $(\mathcal{E} \to \mathcal{X}_{\mathcal{T}}, \nabla)$ defines the universal isomonodromic deformation of $(E_0 \to X_0, \nabla_0)$. For each parameter $\tau \in \mathcal{T}$, this universal object induces the unique (modulo isomorphism) non-singular rank 2 connection $(E_\tau \to X_\tau, \nabla_\tau)$ over X_τ, having monodromy ρ.

b) **Logarithmic case**
Let M be a complex manifold. Given a normal crossing divisor $\mathcal{D}$, the set of integrable logarithmic rank r connections (E, ∇) over M with divisor $\mathcal{D}$ (modulo *meromorphic transformations*) is in one-to-one correspondence wih the set of representations of $\pi_1(M \setminus \mathcal{D})$ in $\mathrm{GL}(r, \mathbf{C})$ (modulo conjugacy), according to a result of P. Deligne [Del70] (see also [Kat76]). Since we study connections modulo *holomorphic transformations*, we will need the more precise version of the Riemann-Hilbert correspondence in rank 2 stated below.[2] We will need the following well-known result, which is a corollary of the Poincaré-Dulac theorem stated for example in [NY04].

Proposition 3.8. *Let (E, ∇) be a flat logarithmic rank 2 connection over a complex manifold M, with local coordinates $(x_1, \ldots, x_N)$, which has a pole in $\{x_1 = 0\}$. Then there are local gauge transformations ϕ such that $\phi^*\nabla$ is given by a system*

$$\mathrm{d}Y = \frac{A}{x_1} Y \mathrm{d}x_1,$$

where the matrix A has one of the following standard forms

$$A = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \begin{pmatrix} \theta + n & 0 \\ 0 & \theta \end{pmatrix} \text{ or } \begin{pmatrix} \theta + n & x_1^n \\ 0 & \theta \end{pmatrix}, \tag{12}$$

with $\theta_1 - \theta_2 \notin \mathbf{Z}$ and $n \in \mathbf{N}$. Moreover, ϕ is unique modulo composition by a matrix of the form respectively

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \lambda & \nu x_1^n \\ 0 & \mu \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & \nu x_1^n \\ 0 & \lambda \end{pmatrix}, \tag{13}$$

where λ, μ, ν are constants.

Remark 3.9. *In particular, a flat logarithmic rank 2 connection $(\mathcal{E}, \nabla)$ on a family of Riemann surfaces is locally constant.*

Since $\exp(\log(x_1) \cdot A)$ is a local fundamental solution, the associated *local monodromies* (that is the conjugacy class of the image under ρ of a small positive loop around $\{x_1 = 0\}$) are respectively

$$M = \begin{pmatrix} e^{2i\pi\theta_1} & 0 \\ 0 & e^{2i\pi\theta_2} \end{pmatrix}, \begin{pmatrix} e^{2i\pi\theta} & 0 \\ 0 & e^{2i\pi\theta} \end{pmatrix} \text{ or } \begin{pmatrix} e^{2i\pi\theta} & 1 \\ 0 & e^{2i\pi\theta} \end{pmatrix}.$$

We call θ_1 and θ_2, respectively θ and $\theta + n$ the *residues* of the connection at the pole $\{x_1 = 0\}$. In the last two cases of (12), we speak about a simple pole *with resonance.* If the local monodromy is a homothecy, that is to say in the second of the cases above, the pole is called a *projectively apparent singularity.*

[2] P. Deligne actually stated a "holomorphic version" of the Riemann-Hilbert correspondence, but he only treated the case where all residues are in the intervall $[0, 1[$. Thereby he excluded the case of projectively apparent singularities.

Remark 3.10. *Note that if a flat rank 2 connection $(\mathcal{E}, \nabla)$ has a projectively apparent singularity at $\{x = 0\}$: $\mathrm{d}Y = \frac{1}{x}\left(\begin{smallmatrix}\theta+n & 0\\ 0 & \theta\end{smallmatrix}\right) Y \mathrm{d}x$, then local monodromy at this singularity is a homothecy and thus projectively trivial. Moreover, the bimeromorphic gauge transformation $\widehat{Y} = \left(\begin{smallmatrix}1 & 0\\ 0 & x^n\end{smallmatrix}\right) Y$ is shifting the second eigenvalue θ of the connection matrix to $\theta + n$ and the resulting Riccati foliation is non-singular at $\{x = 0\}$.*

Let (E, ∇) be a flat logarithmic connection on M. An isomorphism ϕ of ∇ on a chart U of M, given by holomorphic gauge transformations without monodromy, is called a *symmetry* on U, if ϕ conjugates the connection matrix to itself. Let U be a small chart containing a pole $\{x_1 = 0\}$. If ∇ is given on U by a system of standard form as in proposition 3.8, then any symmetry on the punctured chart $U^* = U \setminus \{x_1 = 0\}$ has the form respectively

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \lambda & \nu x_1^n \\ o x^{-n} & \mu \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & \nu x_1^n \\ 0 & \lambda \end{pmatrix}, \tag{14}$$

where λ, μ, ν, o are constants. In the non-projectively apparent case and in the projectively apparent case with $n = 0$, any symmetry on the punctured chart U^* can thus be analytically continued to a symmetry on the unpunctured chart U. We notice further that in the projectively apparent case, a symmetry ϕ on U^* can be continued analytically to U if, and only if, ϕ keeps the *special line* $L \subset \mathbf{C}^2$ invariant over U^*, where L is generated by $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ in the upper coordinates. In the case of tracefree connections, the special line L corresponds to the sub-vector space of bounded local solutions at $\{x_1 = 0\}$.

Let us now come back to the Riemann-Hilbert correspondence. Let $\mathcal{D} = \sum_{i=1}^m \mathcal{D}^i$ be a disjoint union of smooth irreducible and simply connected divisors $\mathcal{D}^i$ of codimension 1 in M. For each $i \in \{1, \ldots, m\}$, fix residues θ_i^1, θ_i^2. Then the monodromy representation establishes the so-called *Riemann-Hilbert map* from the set of integrable logarithmic rank 2 connections (E, ∇) over M with divisor $\mathcal{D}$ and residues θ_i^1, θ_i^2 along $\mathcal{D}^i$ for $i \in \{1, \ldots, m\}$ (modulo holomorphic gauge transformations) to the set of representations ρ from $\pi_1(M^*)$ to $\mathrm{GL}(2, \mathbf{C})$ (modulo conjugacy) such that $\rho(\gamma_i)$ has eigenvalues $e^{2i\pi\theta_i^1}$ and $e^{2i\pi\theta_i^2}$, where γ_i describes a small positive loop around $\mathcal{D}_i$. Here $M^* = M \setminus \mathcal{D}$.

Theorem 3.11 (Riemann-Hilbert correspondence)**.** *Fix $M, \mathcal{D}^i, \gamma_i$ and θ_i^1, θ_i^2 for $i \in \{1, \ldots, m\}$ as above.*

1. *The Riemann-Hilbert map is surjective.*
2. *If for each $i \in \{1, \ldots, m\}$ we have $\theta_i^1 - \theta_i^2 \notin \mathbf{Z}$ or $\theta_i^1 - \theta_i^2 = 0$, then the Riemann-Hilbert map is bijective.*
3. *If for some $i \in \{1, \ldots, m\}$ we have $\theta_i^1 - \theta_i^2 \in \mathbf{Z}^*$, then the restriction of the Riemann-Hilbert map to the set of connections (E, ∇) with no projectively apparent singularities is injective.*

Proof: Let us recall the main steps of the proof, written in more detail in [Bri04]. By the classical Riemann-Hilbert correspondence, we get a non-singular integrable rank r connection $(E^* \to M^*, \nabla^*)$ over the punctured curve M^* having monodromy ρ. Recall that this connection is unique up to isomorphism.

Let $\mathcal{U}^i$ be a germ of neighborhood of $\mathcal{D}^i$ in M. In order to complete the connection (E^*, ∇^*) at the polar set, let us now construct an integrable connection (E^i, ∇^i) on $\mathcal{U}^i$ having a logarithmic pole on $\mathcal{D}^i$ and local monodromy ρ. Let E^i be the trivial vector bundle over $\mathcal{U}^i$. Since $\mathcal{D}^i$ is smooth, it can be covered by local charts with coordinates (t, x) such that $\mathcal{D}^i$ is given by $\{x = 0\}$. In each of these local coordinates we define ∇^i by $\mathrm{d}Y_i = \frac{A_i}{x} Y_i \mathrm{d}x$, where A_i has the standard form (12) corresponding to the prescribed residues θ_1^i, θ_2^i. If $\theta_1^i - \theta_2^i \in \mathbf{Z}$, the local monodromy determines whether we have to choose the projectively apparent or the non-projectively apparent standard form. On the intersection of two such local open sets, the coordinate transformation φ may change the connection matrix, but this conjugacy can be annihilated by a convenient gauge transformation ϕ, according to proposition 3.8. Since $\mathcal{D}^i$ is simply connected, we can choose these gauge-coordinate transformations (φ, ϕ) to glue the local connections into a connection $(E^i \to \mathcal{U}^i, \nabla^i)$, as desired. Again by proposition 3.8, this connection is unique up to isomorphisms on $\mathcal{U}^i$.

Now on the intersection $\mathcal{U}^{i^*} = \mathcal{U}^i \cap M^*$, the connections (E^*, ∇^*) and (E^i, ∇^i) are both non-singular, integrable connections with (local) monodromy ρ. By the classical Riemann-Hilbert correspondence, there is an isomorphism allowing to glue these two connections. In doing so for each $i \in \{1, \ldots, m\}$, we finally obtain a logarithmic integrable connection $(E \to M, \nabla)$ having the prescribed monodromy ρ and the prescribed residues along its poles $\mathcal{D}^i$. In the non-projectively apparent case or if $\theta_1^i = \theta_2^i$, this connection is still unique up to isomorphism. Indeed the gluing is uniquely defined up to a symmetry on the punctured set $\mathcal{U}^{i^*}$, which can be continued anaytically to an isomorphism on the unpunctured set $\mathcal{U}^i$. $\square$

Let us look some closer at the case of projectively apparent singularities. In this case, we cannot associate a unique connection to a given monodromy representation and given residues in general. Indeed, let $\mathcal{D}^{a_j}$ be a projectively apparent singularity with $\theta_{a_j}^1 - \theta_{a_j}^2 \in \mathbf{Z}^*$. Let (E, ∇) be a flat logarithmic connection and let ϕ^{a_j} be the associated gluing between (E^*, ∇^*) and (E^{a_j}, ∇^{a_j}) as in the above proof. We may suppose that ∇^{a_j} is given over $\mathcal{U}^{a_j}$ by a connection matrix in standard form (12). In composing ϕ^{a_j} by symmetries on $\mathcal{U}^{a_j *}$ of the form $\begin{pmatrix} 1 & 0 \\ cx_{a_j}^n & 1 \end{pmatrix}$, we get a smooth family of connections, parametrized by $c \in \mathbf{C}$. Two connections in this family are conjugated over $\mathcal{U}^{a_j *}$ by a symmetry which can not be continued to a gauge-transformation on $\mathcal{U}^{a_j}$. The family of connections with constant monodromy we have obtained is thus non-trivial except for the very special case where every symmetry can be continued analytically to an isomorphism of (E, ∇) in restriction to $X \setminus \mathcal{D}_{a_j}$. In fact this happens only if there is only one singularity, which is the projectively apparent one, and the monodromy is trivial.

Yet we can restore the bijectivity in the Riemann-Hilbert correspondence for general logarithmic connections, if we associate special lines to every monodromy representation for the projectively apparent singularities it contains. More precisely, we first fix for each resonant divisor $\mathcal{D}^j$ simple paths $\delta_j : [0,1] \to M^*$ from the chosen base point z_0 of the fundamental group $\pi_1(M^*, z_0)$ to a point in $\mathcal{D}^j$. We consider pairs (ρ, L), where ρ is a monodromy representation of $\pi_1(M^*, z_0)$ compatible with the fixed residues along $\mathcal{D}^i$ for $i \in \{1, \dots, m\}$. Denote by $(a_1, \dots, a_\eta)$ the indices such that $\mathcal{D}^{a_j}$ is a projectively apparent singularity for ρ and $\theta_1^{a_j} - \theta_2^{a_j} \in \mathbf{Z}^*$. Then $L = (L_{a_1}, \dots, L_{a_\eta})$ shall be a η-tuple of lines $L_{a_j} \subset \mathbf{C}^2$. Now let (E, ∇) be a flat logarithmic connection on M with polar divisor $\mathcal{D}$ and projectively apparent singularities on $\mathcal{D}^{a_1}, \dots, \mathcal{D}^{a_\eta}$. Choose a fundamental solution S of ∇ in a neighborhood of z_0. Then there is a unique η-tuple L of lines $L_{a_1}, \dots, L_{a_\eta}$ in the 2-dimensional vector space generated by S, such that the analytic continuation of S along δ_{a_j} allows to identify L_{a_j} to the special line of ∇ near $\delta_{a_j}(1)$. Now if ρ is just the usual representation of monodromy with respect to our choice of S, then a base change $\widetilde{S} = AS$ with $A \in \mathrm{GL}(2, \mathbf{C})$ acts on (ρ, L) in the following way

$$A \cdot (\rho, L) = (A\rho A^{-1}, AL).$$

The new defined Riemann-Hilbert map onto the set of pairs (ρ, L) as above (modulo the previous action of $\mathrm{GL}(2, \mathbf{C})$) is a bijection.

Remark 3.12. *The notion of monodromy representations with additional information on the position of the special lines should not be confused with the notion of* parabolic connections*, which are connections with an additional structure. Stable parabolic connections on $\mathbf{P}^1$ have a smooth moduli space, but the associated Riemann-Hilbert map is not injective (see [IIS06]).*

Corollary 3.13. *Let $Z \subset M$ be a complex submanifold of M such that the inclusion map $i : Z^* \to M^*$ with $Z^* = Z \cap M^*$ provides an isomorphism $i_* : \pi_1(Z^*, z_0) \xrightarrow{\sim} \pi_1(M^*, z_0)$ with $z_0 \in Z^*$. Assume that $\mathcal{D}$ is transversal to Z and let $\mathcal{D}_Z = \sum_{i=1}^m \mathcal{D}_Z^i$ be the induced divisor on Z.*
Let (E_Z, ∇_Z) be a flat logarithmic rank 2 connection over Z with polar set $\mathcal{D}_Z$ and residues θ_i^1, θ_i^2 along $\mathcal{D}_Z^i$, having monodromy ρ. Then (E_Z, ∇_Z) extends to a flat logarithmic rank 2 connection (E, ∇) over M, with polar set $\mathcal{D}$ and residues θ_i^1, θ_i^2 along $\mathcal{D}^i$, wich is unique in the following sense.
Let $(\widetilde{E}, \widetilde{\nabla}), (E, \nabla)$ be two such connections. Then each isomorphism $\psi_Z : (\widetilde{E}, \widetilde{\nabla})|_Z \xrightarrow{\sim} (E, \nabla)|_Z$ over Z, extends to an isomorphism $\psi : (\widetilde{E}, \widetilde{\nabla}) \xrightarrow{\sim} (E, \nabla)$ over M.

Proof: By the Riemann-Hilbert correspondence there is a unique connection (E, ∇) having the same monodromy representation and the same special lines at the projectively apparent singularities as (E_Z, ∇_Z).
Let $(\widetilde{E}, \widetilde{\nabla})$ and (E, ∇) be as above. By corollary 3.7, we get a unique isomorphism $\psi^* : (\widetilde{E}, \widetilde{\nabla})|_{M^*} \xrightarrow{\sim} E, \nabla)|_{M^*}$ such that $\psi^*|_{Z^*} = \psi_Z|_{Z^*}$.
Now let $\mathcal{U}^i$ be a small neigbhorhood of $\mathcal{D}^i$ in M. Since $(\widetilde{E}, \widetilde{\nabla})$ and (E, ∇) have the same residues along $\mathcal{D}^i$, in appropriate coordinates they are both given by the same

standard connection matrix (12). Now ψ^* defines a symmetry on $\mathcal{U}^{i^*}$ and thus extends holomorphically in the non-projectively apparent case to an isomorphism on $\mathcal{U}^i$, as desired. In the projectively apparent case however, the continuation ψ of ψ^* could a priori be meromorphic. Yet (14) shows that if ψ is not holomorphic on $\mathcal{D}^i$, then it has a pole all along the divisor $\mathcal{D}^i$. This is impossible since $\psi|_Z = \psi_Z$ is holomorphic in D^i_Z. □

If ∇_0 is logarithmic, this corollary provides a unique logarithmic, flat and locally trivial rank 2 connection $(\mathcal{E} \to \mathcal{X}, \nabla)$ with polar set $\mathcal{D}_{\mathcal{T}}$ over the universal Teichmüller curve $(\mathcal{X}_{\mathcal{T}}, \mathcal{D}_{\mathcal{T}})$, such that

$$(E_{\tau_0} \to X_{\tau_0}, \nabla_{\tau_0}) = (E_0 \to X_0, \nabla_0).$$

This defines the universal isomonodromic deformation in the logarithmic case. This universal object induces for each parameter $\tau \in \mathcal{T}$ the unique connection (E_τ, ∇_τ) with simple poles in D_τ which realizes the monodromy, the residues and the special lines for each projectively apparent singularity, all prescribed by the initial connection $(E_0 \to X_0, \nabla_0)$.

Remark 3.14. *For this construction of universal isomonodromic deformations of logarithmic connections with the help of the Riemann-Hilbert correspondence, we are using the method of B. Malgrange developped in [Mal83a]. It is also possible to use the local method developped in [Mal86], which generalizes easily to a global construction in the logarithmic case. This method avoids the Riemann-Hilbert correspondence and allows to obtain directly corollary 3.13.*

c) **General case**

We are now going to construct the universal isomonodromic deformation in the case of multiple poles with methods similar to those developped by B. Malgrange in [Mal83a], [Mal83b] and [Mal86] for the genus 0-case (see also [Pal99] and [Kri02]). Like in the logarithmic case, we get a non-singular connection $(\mathcal{E}^*, \nabla^*)$ over the punctured universal curve $\mathcal{X}^*_{\mathcal{T}}$ over the Teichmüller space $\mathcal{T} = \mathrm{Teich}(g, m)$. We may also construct local connections $(\mathcal{E}^i, \nabla^i)$ in order to stuff the gaps. But since the gluing will not be unique in general, we get additional parameters in the construction.

More explicitely, let $(\mathcal{E}^* \to \mathcal{X}^*_{\mathcal{T}}, \nabla^*)$ be the non-singular integrable connection on the punctured Teichmüller curve extending $(E_0 \to X_0, \nabla_0)|_{X_0^*}$ with $X_0^* = X_0 \setminus D_0$.

From the Bers construction of the universal Teichmüller curve (see [Hub06]) follows the existence of tubular neighborhoods $\mathcal{U}^i_{\mathcal{T}}$ of $\mathcal{D}^i_{\mathcal{T}}$. In other words, there are global coordinates of $\mathcal{D}^i_{\mathcal{T}}$ in $\mathcal{U}^i_{\mathcal{T}}$, *i.e.* holomorphic functions $\xi_i : \mathcal{U}^i_{\mathcal{T}} \to \mathbf{C}$ such that $\mathrm{div}(\xi_i) = \mathcal{D}^i_{\mathcal{T}}$. We may suppose that the function induced by ξ_i on $\mathcal{U}^i_{\mathcal{T}} \cap X_0$ is the coordinate function x_i on $U^i_0 = \mathcal{U}^i_{\mathcal{T}} \cap X_0$. We will define a local connection ∇^i on the trivial bundle $\mathcal{E}^i$ over the germification of such a tubular neighborhood $\mathcal{U}^i_{\mathcal{T}}$. If ∇_0 is defined by a system $\mathrm{dY} = A(x_i)Y\mathrm{d}x_i$ on U^i_0, which has a pole in $\{x_i = 0\}$, then ∇^i shall be given on $\mathcal{U}^i_{\mathcal{T}} \times \mathbf{C}^2$ by the product connection $\mathrm{dY} = A(\xi_i)Y\mathrm{d}\xi_i$, having a pole in $\mathcal{D}^i_{\mathcal{T}}$.

According to the classical Riemann-Hilbert correspondence, there is an isomorphism gluing ∇^i and ∇^* over the punctured chart ${\mathcal{U}^i_{\mathcal{T}}}^* = \mathcal{U}^i_{\mathcal{T}} \setminus \mathcal{D}^i$. Let us now see in detail why this gluing might not be unique. We choose appropriate local coordinates, where the connections $(\mathcal{E}^i, \nabla^i)$ and $(\mathcal{E}^*, \nabla^*)$ are constant. Let (τ, x_i) be a local coordinate on ${\mathcal{U}^i_{\mathcal{T}}}^*$. Let $\mathcal{V} \times \mathbf{C}^2$ be a simply connected subchart of ${\mathcal{U}^i_{\mathcal{T}}}^* \times \mathbf{C}^2$ where the connection ∇^* is given by the system $\mathrm{d}Y = 0$. The connection ∇^i will still be given by $\mathrm{dY} = A(x_i)Y\mathrm{d}x_i$. Let now $Y \mapsto \phi(x_i)Y$ be the gauge transformation on $\mathcal{V} \times \mathbf{C}^2$ defined by the restriction of the gluing isomorphism between the two connections. Let φ be a coordinate change on $\mathcal{U}^i_{\mathcal{T}}$ fixing $\mathcal{D}^i_{\mathcal{T}}$. In our coordinates, φ is given by a holomorphic family $(\varphi_\tau(x_i))$ of holomorphic diffeomorphisms of the germ $(\mathbf{C}, 0)$. Now for each parameter τ, the gauge-coordinate transformation $(\varphi_\tau(x_i), \phi(x_i)Y)$ will also conjugate the two systems. We thereby define another gluing isomorphism on ${\mathcal{U}^i_{\mathcal{T}}}^*$. In other

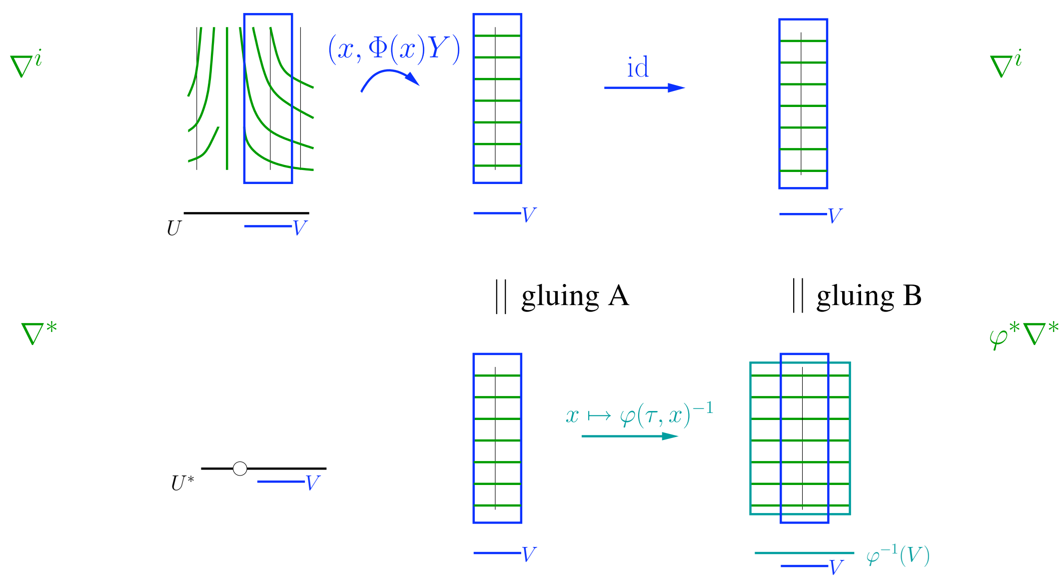

Figure 3: Non-unicity in the gluing construction, due to diffeomorphisms of the base curve

words, it may be sufficient to choose another gluing between $\mathcal{U}^i_{\mathcal{T}}$ and the punctured base curve $\mathcal{X}^*_{\mathcal{T}}$ to get another flat connection $(\mathcal{E}, \nabla)$, which extends (E_0, ∇_0) if $\varphi_{\tau_0} = \mathrm{id}$.

We will now examine under which conditions this second gluing defines the same connection as the initial one. In order to simplify notations, let us consider the gluing between $\varphi^*(\mathcal{E}^i, \nabla^i)$ and $(\mathcal{E}^*, \nabla^*)$ in restriction to a fixed parameter. We will get the same connection for this second gluing if, and only if, $\varphi^*(\mathcal{E}^i, \nabla^i)$ given by the system $\mathrm{dY} = A \circ \varphi^{-1}(x_i) Y \mathrm{d}\varphi^{-1}(x_i)$ over $\mathcal{V}$, is conjugated to $(\mathcal{E}^i, \nabla^i)$ over $\mathcal{V}$ by a holomorphic gauge transformation $Y \mapsto \widetilde{\phi}(x_i)Y$.

$$
\begin{tikzcd}
(\mathcal{E}^*, \nabla^*) & & \varphi^*(\mathcal{E}^*, \nabla^*) \\
(\mathcal{E}^i, \nabla^i) \arrow[u, "{(x_i,\phi(x_i)Y)}"] \arrow[urr, "{(\varphi^{-1}(x_i),\phi(x_i)Y)}"] & & \varphi^*(\mathcal{E}^i, \nabla^i) \arrow[u, "{(x_i,\phi(x_i)Y)}"'] \arrow[ll, dashed, "{(x_i,\widetilde{\phi}(x_i)Y)}"']
\end{tikzcd}
$$

Or equivalently, if there is a holomorphic gauge transformation $\widetilde{\phi}(x_i)$ such that $(\varphi^{-1}(x_i), \widetilde{\phi}(\varphi^{-1}(x_i))Y)$ is conjugating $(\mathcal{E}^i, \nabla^i)$ to itself.

$$
\begin{tikzcd}
(\mathcal{E}^i, \nabla^i) & & \varphi^*(\mathcal{E}^i, \nabla^i) \arrow[ll, dashed, "{(x_i,\widetilde{\phi}(x_i)Y)}"'] \arrow[dl, "\varphi"] \\
& (\mathcal{E}^i, \nabla^i) \arrow[ul, "{(\varphi^{-1}(x_i),\widetilde{\phi}(\varphi^{-1}(x_i))Y)}"] &
\end{tikzcd}
$$

We have already seen that there is a unique connection resulting from such a gluing in the logarithmic case. In the general case, we have the following result.

Lemma 3.15. *Let ∇^i be a connection on the trivial vector bundle E^i over a germ $(\mathbf{C}, 0)$ with coordinate x, having a pole of order l at $\{x = 0\}$. Let $\varphi(x)$ be a holomorphic diffeomorphism of $(\mathbf{C}, 0)$ such that*

$$\varphi(x) \equiv \mathrm{id}(x) \mod x^l.$$

Then there is a holomorphic gauge transformation $\widetilde{\phi}(x)$ such that $(\varphi(x), \widetilde{\phi}(\varphi(x))Y)$ is conjugating (E^i, ∇^i) to itself.

Proof: On a trivialization chart, (E^i, ∇^i) is given by $\mathrm{d}Y = A(x)Y\mathrm{d}x$ with connection matrix

$$A(x) = \frac{1}{x^l}\begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix}.$$

Since A has a pole of order l, any holomorphic vector field on E^i tangent to the connection ∇^i is a holomorphic multiple of

$$\mathcal{W} = x^l \frac{\partial}{\partial x} + (a(x)y_1 + b(x)y_2)\frac{\partial}{\partial y_1} + (c(x)y_1 - a(x)y_2)\frac{\partial}{\partial y_2}.$$

Now $(\varphi_s(x))_{s\in[0,1]}$ with $\varphi_s(x) = s\varphi(x) + (1-s)\mathrm{id}(x)$ defines an analytic isotopy of holomorphic diffeomorphisms joining φ to the identity. It defines the flow of an analytic isotopy of vector fields

$$v_s(x) = \frac{\partial}{\partial s} + \varphi_s^* v_0(x),$$

where $v_0(x) = (\varphi(x) - x)\frac{\partial}{\partial x}$. Now v_0 has a zero of order l at $x = 0$. Then $v_s = \frac{\partial}{\partial s} + x^l f_s(x)\frac{\partial}{\partial x}$, where (f_s) is an analytic isotopy of holomorphic maps. We can lift v_s to the following vector field $\mathcal{V}_s$ on E^i tangent to the connection ∇^i:

$$\mathcal{V}_s = \frac{\partial}{\partial s} + f_s(x)\mathcal{W}.$$

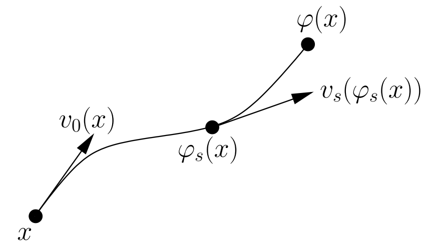

Figure 4: Family of vector fields associated to the family of diffeomorphisms

The flow of $\mathcal{V}_s$ defines an analytic isotopy $(\Phi_s^{\mathcal{V}})_{s\in[0,1]}$ of holomorphic gauge-coordinate-transformations. It is indeed a gauge-transformation with respect to the Y-coordinate, since we are integrating in the Lie-algebra of gauge-transformations. By construction, the holomorphic gauge-coordinate-transformation $\Phi_1^{\mathcal{V}}$ on E^i keeps ∇^i invariant and is of the form

$$\Phi_1^{\mathcal{V}}(x,Y) = (\varphi(x), \widetilde{\widetilde{\phi}}(x)Y) = (\varphi(x), \widetilde{\phi}(\varphi(x))Y)$$

□

Inversely, if there is an analytic isotopy $(\varphi_s(x), \phi_s(x)Y)_{s\in[0,1]}$ with $(\varphi_0, \phi_0) = (\mathrm{id}, I)$ of holomorphic gauge-coordinate-transformations on E^i keeping ∇^i invariant, then we can associate a family of holomorphic vector fields tangent to the connection, as we will see in lemma 3.19. This implies $\varphi_s(x) \equiv \mathrm{id}(x) \mod x^l$.

It is possible that there are gauge-coordinate transformations on E^i keeping ∇^i invariant and such that the induced coordinate change is not tangent to the identity. For example $\mathrm{d}Y = \frac{1}{x^l} Y \mathrm{d}x$ is invariant under the coordinate transformation $x \mapsto e^{\frac{2i\pi}{l-1}} x$. But the set of such coordinate transformations is discrete.

In order to construct the universal isomonodromic deformation in the case of multiple poles, consider for each pole of order $n_i > 0$ the set $J^i = \mathrm{Jets}^{<n_i}(\mathrm{Diff}(\mathbf{C}, 0))$ of $(n_i - 1)$-jets of biholomorphisms of $(\mathbf{C}, 0)$. We identify J^i to $\mathbf{C}^* \times \mathbf{C}^{n_i - 2}$, where $s = (s_1, \ldots, s_{n_i-1})$ is associated to $\varphi_s(x) = s_1 x + s_2 x^2 + \ldots + s_{n_i-1} x^{n_i-1}$. For a simple pole $x_j = 0$ the set J^j is reduced to the identity-singleton.
Let J be the universal cover

$$J = \widetilde{J^1} \times \ldots \times \widetilde{J^m}$$

of the space of jets, where $\widetilde{J^i} = \widetilde{\mathbf{C}^*} \times \mathbf{C}^{n_i-2}$. Our parameter space T for the universal isomonodromic deformation will be

$$T = J \times \mathcal{T}.$$

Remark 3.16. *We consider a simply connected parameter space in order to avoid monodromy phenomena along the parameter space.*

Our universal curve $(\mathcal{X}, \mathcal{D}) \to T$ shall be the product of the the space of jets J with the universal Teichmüller curve :

$$(\mathcal{X}, \mathcal{D}) = (J \times \mathcal{X}_{\mathcal{T}}, J \times \mathcal{D}_{\mathcal{T}}).$$

This curve will be the base curve of the universal isomonodromic deformation of $(E_0 \to X_0, \nabla_0)$.

As before, denote by $\mathcal{X}^*$ the universal curve minus the distinguished points and their deformations $\mathcal{D}^i = J \times \mathcal{D}^i_{\mathcal{T}}$. Let $(\mathcal{E}^*, \nabla^*)$ be the unique non-singular integrable connection over $\mathcal{X}^*$ having monodromy ρ. For $t_0 = ((\mathrm{id}, \ldots, \mathrm{id}), \tau_0) \in T$, we have

$$(\mathcal{E}^*, \nabla^*)|_{t=t_0} = (E_0^*, \nabla_0^*).$$

We have again tubular neighborhoods $\mathcal{U}^i = J \times \mathcal{U}^i_{\mathcal{T}}$ of $\mathcal{D}^i$ on $\mathcal{X}$ and functions $\xi_i : \mathcal{U}^i \to \mathbf{C}$, identical to the Teichmüller coordinate, but seen on the bigger space, satisfying

$$\mathcal{D}^i = \mathrm{div}(\xi_i).$$

On $U_0^i = \mathcal{U}^i|_{t=t_0}$ with coordinate x_i, the initial connection induces a local connection (E_0^i, ∇_0^i), defined by a system

$$\mathrm{d}Y = A(x_i) Y \mathrm{d}x_i$$

on the trivial vector bundle E_0^i.

Then on the tubular neighborhood $\mathcal{U}^i$, we define an integrable, locally constant connection $(\mathcal{E}^i, \nabla^i)$ with polar set $\mathcal{D}^i$. Firstly, we define the connection

$$\widetilde{\nabla}_0^i : \quad \mathrm{d}Y = A(\xi_i) Y \mathrm{d}\xi_i$$

on the trivial vector bundle $\widetilde{E}_0^i$ over $\mathcal{U}^i = J \times \mathcal{U}^i_{\mathcal{T}}$ as a product from the initial connection (E_0^i, ∇_0^i). Then for coordinates $(\varphi_1, \ldots, \varphi_m) \in J = \prod_{i=1}^m \widetilde{J}^i$, we define ∇^i on the trivial vector bundle $\mathcal{E}^i$ by

$$\nabla^i = \varphi_i^* \widetilde{\nabla}_0^i$$

on $U^i = J \times \mathcal{U}^i_{\mathcal{T}}$. Namely on $\mathcal{U}^i$, we define

$$\nabla^i : \quad \mathrm{d}Y = A((\varphi_i)^{-1}(\xi_i)) Y \mathrm{d}((\varphi_i)^{-1}(\xi_i)).$$

On the trivial vector bundle E_0^i, we then have

$$\nabla_0^i = \nabla^i|_{t=t_0}.$$

We now have to glue $(\mathcal{E}^i, \nabla^i)$ to $(\mathcal{E}^*, \nabla^*)$. Let ψ_0^i be the gluing isomorphism from (E_0^*, ∇_0^*) to (E_0^i, ∇_0^i).

$$\begin{tikzcd}
(E_0^*, \nabla_0^*) \arrow[r, "\psi_0^i"] \arrow[d, equal] & (E_0^i, \nabla_0^i) \arrow[d, equal] \\
(\mathcal{E}^*, \nabla^*)|_{t=t_0} & (\mathcal{E}^i, \nabla^i)|_{t=t_0}
\end{tikzcd}$$

On the base curve, we choose the natural, *i.e.* the identity gluing from $\mathcal{X}^*$ to $\mathcal{U}^i$ on the intersection $\mathcal{U}^{i^*} = \mathcal{U}^i \backslash \mathcal{D}^i$. According to the classical Riemann-Hilbert correspondence and its corollary, there is an isomorphism ψ^i given by gauge transformations over $\mathcal{U}^{i^*}$, which is gluing $(\mathcal{E}^*, \nabla^*)$ to $(\mathcal{E}^i, \nabla^i)$ and which extends the initial gluing isomorphism :

$$\psi^i|_{t=t_0} = \psi^i_0.$$

Gluing in this manner each of the local connections $(\mathcal{E}^*, \nabla^*)$ to $(\mathcal{E}^i, \nabla^i)$, we constructed a flat integrable, locally constant tracefree connection $(\mathcal{E} \to \mathcal{X}, \nabla)$ over $\mathcal{X} \to T$ with polar set $\mathcal{D}$, satisfying

$$(\mathcal{E} \to \mathcal{X}, \nabla)|_{t=t_0} = (E_0 \to X_0, \nabla_0).$$

We remark that the so-defined universal isomonodromic deformation is global in reference to the Teichmüller space as well as in reference to the space of jets.

If the Teichmüller space $\mathcal{T}$ has dimension $3g - 3 + m \geq 0$, then the dimension of the parameter space T of the universal isomonodromic deformation constructed above is $3g - 3 + m + \sum_{i=1}^m (n_i - 1)$. Thus

$$\dim(T) = 3g - 3 + n.$$

3.2 Universal Property

Let us now prove the universal property theorem 3.2 for the above constructed universal isomonodromic deformation $(\mathcal{E} \to \mathcal{X}, \nabla)$. Let $(\widetilde{\mathcal{E}} \to \widetilde{X}, \widetilde{\nabla})$ be another isomonodromic deformation of the initial connection. Let n (resp. m) be the number of poles counted with (resp. without) multiplicity, as before. We will denote by $\sum_{i=1}^m n_i \widetilde{\mathcal{D}}^i$ (resp. $\widetilde{\mathcal{D}}$) the divisor (resp. the reduced divisor) of $(\widetilde{\mathcal{E}} \to \widetilde{X}, \widetilde{\nabla})$. The reduced divisor of ∇_0 will be denoted by D_0. Using the product structure of the parameter space $T = J \times \mathcal{T}$, we will construct holomorphic maps (f, F) extending the initial isomorphism (f_0, F_0) of marked Riemann surfaces, such that the diagram

$$\begin{tikzcd} \widetilde{\mathcal{X}} \arrow[r, "F"] \arrow[d] & \mathcal{X} \arrow[d] \\ \widetilde{T} \arrow[r, "f"] & T \end{tikzcd}$$

commutes and such that, in restriction to each parameter $\tilde{t} \in \widetilde{T}$, the map F induces an isomorphism of marked Riemann surfaces. Therefore we will firstly consider the universal curve respective to the deformation in the Teichmüller space, secondly the deformation respective to the jets. Then we will define an isomorphism $\psi : (\widetilde{\mathcal{E}}, \widetilde{\nabla}) \xrightarrow{\sim} F^*(\mathcal{E}, \nabla)$, which extends the given isomorphism $\psi_0 : (\widetilde{E_0}, \widetilde{\nabla}_0) \xrightarrow{\sim} F_0^*(E_0, \nabla_0)$. Here ψ (resp. ψ_0) are isomorphisms between connections on the same base curve $\widetilde{\mathcal{X}}$, (resp. $\widetilde{X}_0$) and will thus be given by local gauge transformations. The maps Ψ (resp. Ψ_0) are then obtained via F (resp. F_0). The triple (f, F, Ψ) will be unique if (g, m) is different from $(0,0), (0,1), (0,2)$ and $(1,0)$. Finally we will study the default of unicity in the special cases.

1. (a) Denote again by $\mathcal{X}_\mathcal{T} \to \mathcal{T}$ the universal Teichmüller curve respective to X_0, and by $\mathcal{X} \to T$ the universal curve underlying the universal isomonodromic deformation of $(E_0 \to X_0, \nabla_0)$. Consider the Teichmüller classifying map h from $(\widetilde{T}, \tilde{t}_0)$ to $(\mathcal{T}, \tau_0)$. The map h is holomorphic and induces a holomorphic map H, making the following diagramm commute

$$\begin{tikzcd} \widetilde{\mathcal{X}} \arrow[r, "H"] \arrow[d] & \mathcal{X}_\mathcal{T} \arrow[d] \\ \widetilde{T} \arrow[r, "h"] & \mathcal{T} \end{tikzcd}$$

and such that H defines an isomorphism of marked Riemann surfaces in each fibre (see [Nag88], page 349).

Remark 3.17. *The cases $g = 0$, $m = 1, 2$ are not explicitly treated in [Nag88], but the result remains true in our context since any analytic fibre space whose fibres are all holomorphically equivalent to one compact connected complex manifold is locally trivial (see [FG65]).*

(b) From the tubular neighborhoods in the Teichmüller curve, via F we get tubular neighborhoods $\widetilde{\mathcal{U}}^i$ of $\widetilde{\mathcal{D}}^i$ and thus the connection $(\widetilde{\mathcal{E}}_0^i, \widetilde{\nabla}_0^i)$, induced by the initial connection $(\widetilde{E}_0, \widetilde{\nabla}_0)$ on $\widetilde{\mathcal{U}}^i \cap X_0$, can be considered as a product connection on $\widetilde{\mathcal{U}}^i$. We will now construct a classifying map in the space of local jets. From the point of view of the product connection $(\widetilde{E}_0^i, \widetilde{\nabla}_0^i)$, the gluing isomorphism between $(\widetilde{\mathcal{E}}^i, \widetilde{\nabla}^i)$ and $(\widetilde{\mathcal{E}}^*, \widetilde{\nabla}^*) = (\widetilde{\mathcal{E}}, \widetilde{\nabla})|_{\widetilde{X}^*}$ will only depend on the gluing of $\widetilde{\mathcal{U}}^i$ to $\widetilde{X}^*$ in the base curve. From now on, we will forget the variable Y in the sense that we only indicate whether or not there are gauge-transformations conjugating two given connections.

Let $(\tilde{t}, x_i)$ be a local coordinate on $\widetilde{\mathcal{U}}^i$ defined in a small neighborhood of $\tilde{t}_0$. Assume that $\widetilde{D}^i$ is given in these coordinates by $\{x_i = 0\}$.

Lemma 3.18. *There is coordinate-transformation $(\tilde{t}, x_i) \mapsto (\tilde{t}, \tilde{\varphi}_i(\tilde{t}, x_i))$ fixing $\widetilde{D}^i$ such that locally on the considered open set of $\widetilde{\mathcal{U}}^i$, we have*

$$\begin{cases} \mathrm{id} = \tilde{\varphi}_i|_{\tilde{t}=\tilde{t}_0} \\ \widetilde{\nabla}^i = (\tilde{\varphi}_i)^* \widetilde{\nabla}_0^i, \end{cases} \tag{15}$$

where $\widetilde{\nabla}_0^i$ is considered as a connection on the trivial bundle $\widetilde{\mathcal{E}}_0^i \times \widetilde{T}$ over $\widetilde{\mathcal{U}}^i$.

Proof: According to the local constancy, there are local gauge-coordinate-transformations

$$(\tilde{t}, x_i, Y_i) \mapsto (\tilde{t}, \varphi_i(\tilde{t}, x_i), \phi_i(\tilde{t}, x_i) \cdot Y_i)$$

trivializing the connection $\widetilde{\nabla}^i$ in the parameter $\tilde{t}$ and fixing $\widetilde{\mathcal{D}}_i$. Then

$$(\tilde{t}, x_i, Y_i) \mapsto (\tilde{t}, (\varphi_i)^{-1}(\tilde{t}_0, \varphi_i(\tilde{t}, x_i)), (\phi_i)^{-1}(\tilde{t}_0, x_i) \cdot \phi_i(\tilde{t}, x_i) \cdot Y_i)$$

is satisfying the conditions (15). $\square$

Now let us consider $\tilde{\varphi}_i(\tilde{t}, x_i)$ as a holomorphic family of holomorphic diffeomorphisms $\tilde{\varphi}_i^{\tilde{t}}(x_i)$ of $(\mathbf{C}, 0)$.

Lemma 3.19. *Let $\tilde{\varphi}_i^{\tilde{t}}$ and $\tilde{\tilde{\varphi}}_i^{\tilde{t}}$ be two holomorphic families of biholomorphisms satisfying (15). Then they are equivalent modulo $x_i^{n_i}$.*

Proof: Consider the biholomorphism $\varphi_{\tilde{t}} = (\tilde{\tilde{\varphi}}_i^{\tilde{t}})^{-1} \circ \tilde{\varphi}_i^{\tilde{t}}$. Then $\varphi_{\tilde{t}}^* \widetilde{\nabla}_0^i = \widetilde{\nabla}_0^i$ for each parameter $\tilde{t} \in \widetilde{T}$. This means there is a gauge transformation $Y_i \mapsto \phi_{\tilde{t}}(x_i) \cdot Y_i$ such that the gauge-coordinate-transformation $(\varphi_{\tilde{t}}, \phi_{\tilde{t}})$ is conjugating the system

$$\mathrm{d}Y_i = \frac{1}{x_i^{n_i}} A(x_i) Y_i \mathrm{d}x_i$$

defining $\widetilde{\nabla}_0^i$, to itself.

Let us now fix a parameter $\tilde{t}_1 \in \widetilde{T}$. Let $\gamma : [0,1] \to \widetilde{T}$ be an analytic path with $\gamma(0) = \tilde{t}_0$, $\gamma(1) = \tilde{t}_1$. Now $(\varphi_{\gamma(s)}, \phi_{\gamma(s)})_{s \in [0,1]}$ is an analytic isotopy of gauge-coordinate-transformations keeping $\widetilde{\nabla}_0^i$ invariant. Moreover, this isotopy contains (id, I) for the initial parameter $s = 0$. As in lemma 3.15, we may associate to this isotopy the analytic family

$$\mathcal{V}_s = \left[\frac{\partial}{\partial s} (\varphi_{\gamma(s)}, \phi_{\gamma(s)}) \right]$$

of vector fields tangent to the connection. We may also consider the analytic family $v_s \circ \varphi_{\gamma(s)}(x_i) = \left[\frac{\partial}{\partial s} \varphi_{\gamma(s)}(x_i) \right]$ of vector fields on the base curve. By construction, each of these vector fields v_s lifts to a holomorphic vector field $\mathcal{V}_s = v_s + f(s, x_i) A Y_i$ tangent to the connection $\widetilde{\nabla}_0^i$.

$$\begin{tikzcd}
(\varphi_{\gamma(s)})_{s \in [0,1]} \arrow[r, rightsquigarrow] & (v_s)_{s \in [0,1]} \arrow[d, rightsquigarrow] \\
(\varphi_{\gamma(s)}, \phi_{\gamma(s)})_{s \in [0,1]} \arrow[r, rightsquigarrow] & (\mathcal{V}_s)_{s \in [0,1]}
\end{tikzcd}$$

By consequence, $v_s \circ \varphi_{\gamma(s)}(x_i)$ has to be zero modulo $x_i^{n_i}$ for each $s \in [0,1]$. Since v_s is analytic and v_0 is the identity, it follows from the construction that $\varphi_{\gamma(s)}$ is equal to the identity modulo $x_i^{n_i}$. In particular,

$$\varphi_{\tilde{t}_1} = \mathrm{id} \mod x_i^{n_i}.$$

□

Inversely, if $\tilde{\varphi}_i$ is satisfying (15) and $\tilde{\tilde{\varphi}}_i$ is a holomorphic family of diffeomorphisms with

$$\tilde{\tilde{\varphi}}_i \equiv \tilde{\varphi}_i \mod x_i^{n_i},$$

then $\tilde{\tilde{\varphi}}_i$ also satisfies (15), according to lemma 3.15.

For each parameter $\tilde{t}$ in a neighborhood W_0 of the initial parameter $\tilde{t}_0$ in $\widetilde{T}$, we can find a biholomorphism $\tilde{\varphi}_i(\tilde{t}, x_i)$ as in lemma 3.18, whose $(n_i - 1)$-jet is uniquely defined according to lemma 3.19. The map associating to a parameter in $\widetilde{T}$ the associated (n_i-1)-jet of diffeomorphisms can be analytically continued along any path in $\widetilde{T}$. Indeed, choose an open set W_1 in the germ $\widetilde{\mathcal{U}}^i$, such that the connection $\widetilde{\nabla}^i$ is locally trivial on this open set up to a convenient gauge-coordinate transformation and such that $W_0 \cap W_1 \neq \emptyset$. Choose a parameter $t_1 \in W_0 \cap W_1$, and let $\tilde{\varphi}_{t_1}$ be the associated diffeomorphism. Recall that $\widetilde{\nabla}^i_0$ can be seen naturally as a connection on $\widetilde{\mathcal{U}}^i$. Like in the above lemmas, we see that on W_1, there is a family of diffeomorphisms $\tilde{\varphi}_i$ such that

$$\begin{cases} \tilde{\varphi}_{t_1} = \tilde{\varphi}_i|_{\tilde{t}=\tilde{t}_1} \\ \widetilde{\nabla}^i = (\tilde{\varphi}_i)^* \widetilde{\nabla}^i_0, \end{cases} \tag{16}$$

and the (n_i-1)-jet of this family is unique. In particular, it continues analytically the family of diffeomorphisms on W_0. Since $\widetilde{T}$ is simply connected, we can associate, by analytic continuation, to each parameter $\tilde{t} \in \widetilde{T}$ a biholomorphism $\tilde{\varphi}_i(\tilde{t}, x_i)$, such that the following map is well-defined and holomorphic:

$$g^i : \begin{array}{ccc} \widetilde{T} & \longrightarrow & J^i \\ \tilde{\varphi}_i & \longmapsto & \tilde{\varphi}_i \mod x_i^{n_i} \end{array} .$$

The map

$$g : (\widetilde{T}, \tilde{t}_0) \xrightarrow{(g^1,\ldots,g^m)} (J^1 \times \ldots \times J^m, (\mathrm{id}, \ldots, \mathrm{id}))$$

constructed in that way can be lifted to the universal cover $J = \widetilde{J^1} \times \ldots \times \widetilde{J^m}$. Finally, denote by G the trivial lift

$$\begin{tikzcd} \widetilde{\mathcal{X}} \arrow[r, "G"] \arrow[d] & J \arrow[d] \\ \widetilde{T} \arrow[r, "g"] & J, \end{tikzcd}$$

mapping fibres to singletons.

Since $T = J \times \mathcal{T}$, we obtain a canonical holomorphic mapping

$$f : (\widetilde{T}, \tilde{t}_0) \xrightarrow{(g,h)} (T, t_0) ,$$

where $t_0 = ((\mathrm{id}, \ldots, \mathrm{id}), \tau_0)$. Recall that the universal curve $\mathcal{X}$ is constructed as a product $\mathcal{X} = J \times \mathcal{X}_{\mathcal{T}}$. The Teichmüller map $H : \widetilde{\mathcal{X}} \to \mathcal{X}_{\mathcal{T}}$ thus lifts via G to a holomorphic map $F : \widetilde{\mathcal{X}} \to \mathcal{X}$, such that

$$\begin{tikzcd} \widetilde{\mathcal{X}} \arrow[r, "F=(G,H)"] \arrow[d] & \mathcal{X} \arrow[d] \\ \widetilde{T} \arrow[r, "f=(g,h)"] & T \end{tikzcd}$$

commutes and F defines an isomorphism of marked Riemann surfaces in each fibre. Moreover, the maps (f, F) are extending (f_0, F_0), by construction.

2. We want to extend the initial isomorphism $\psi_0 : (\widetilde{E}_0, \widetilde{\nabla}_0) \to F_0^*(E_0, \nabla_0)$ to an isomorphism $\psi : (\widetilde{\mathcal{E}}, \widetilde{\nabla}) \to F^*(\mathcal{E}, \nabla)$, both given by gauge transformations in appropriate coordinates of the common base curve $\widetilde{X}_0$, respectively $\widetilde{\mathcal{X}}$. Therefore we decompose ψ_0 into an isomorphism $\psi_0^* : (\widetilde{E}_0^*, \widetilde{\nabla}_0^*) \to F_0^*(E_0^*, \nabla_0^*)$ on the punctured base curve $\widetilde{X}_0^* = \widetilde{X}_0 \setminus \widetilde{D}_0$, and isomorphisms ψ_0^i defined in neighborhoods of the poles via trivial gluing isomorphisms $\widetilde{\Phi}_0^i = (\mathrm{id}, I)$ respectively $\Phi_0^i = (\mathrm{id}, I)$:

$$
\begin{tikzcd}
F_0^*(E_0, \nabla_0) & & F_0^*(E_0^*, \nabla_0^*) \arrow[r, "\Phi_0^i"] & F_0^*(E_0^i, \nabla_0^i) \\
& \arrow[r, squiggly] & & \\
(\widetilde{E}_0, \widetilde{\nabla}_0) \arrow[uu, "\psi_0"] & & (\widetilde{E}_0^*, \widetilde{\nabla}_0^*) \arrow[uu, "\psi_0^*"] \arrow[r, "\widetilde{\Phi}_0^i"'] & (\widetilde{E}_0^i, \widetilde{\nabla}_0^i) \arrow[uu, "\Psi_0^i"']
\end{tikzcd}
$$

We may decompose $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ (resp. $F^*(\mathcal{E}, \nabla)$) into the connections $(\widetilde{\mathcal{E}}^*, \widetilde{\nabla}^*)$ (resp. $F^*(\mathcal{E}^*, \nabla^*)$) induced on the punctured base curve $\widetilde{\mathcal{X}}^* = \widetilde{\mathcal{X}} \setminus \mathcal{D}$, and local connections $(\widetilde{\mathcal{E}}^i, \widetilde{\nabla}^i)$ (resp. $F^*(\mathcal{E}^i, \nabla^i)$) on $\widetilde{\mathcal{U}}^i$, together with gluing isomorphisms $\widetilde{\Phi}_i = (\tilde{\varphi}_i, \widetilde{\phi}_i)$ (resp. $\Phi_i = (\varphi_i, \phi_i)$). According to lemma 3.18 we may suppose that in a small neighborhood of the initial parameter in $\mathcal{U}^i$ respectively $\widetilde{\mathcal{U}}^i$ we have

$$
\begin{cases}
(\tilde{\varphi}_i, \widetilde{\phi}_i)|_{\tilde{t}=\tilde{t}_0} = (\mathrm{id}, I) & \text{resp.} \quad (\varphi_i, \phi_i)|_{t=t_0} = (\mathrm{id}, I) \\
\widetilde{\nabla}^i = \widetilde{\nabla}_0^i & \text{resp.} \quad F^*(\nabla^i) = F_0^* \nabla_0^i,
\end{cases}
$$

where $\widetilde{\nabla}_0^i$ (resp. $F_0^* \nabla_0^i$) are seen on the vector bundle $\widetilde{E}_0^i \times \widetilde{T}$ (resp. $F_0^*(E_0^i) \times \widetilde{T}$) over $\widetilde{\mathcal{U}}^i$. Moreover, we may suppose by lemma 3.15 that $\tilde{\varphi}_i$ and φ_i are $(n_i - 1)$-jets of diffeomorphisms with respect to the same coordinates on the base curve. In the new coordinates, the connection matrices of $\widetilde{\nabla}^i$ and $F^*(\nabla^i)$ do not depend on the parameter. Thus ψ_0^i extends trivially into an isomorphism

$$
\begin{cases}
\psi^i : (\widetilde{\mathcal{E}}^i, \widetilde{\nabla}^i) \xrightarrow{\sim} F^*(\mathcal{E}^i, \nabla^i) \\
\psi^i|_{\tilde{t}=\tilde{t}_0} = \psi_0^i,
\end{cases}
\tag{17}
$$

given by gauge-coordinate transformations with respect to the coordinate transformation $\varphi_i \circ (\tilde{\varphi}_i)^{-1}$. By lemma 3.19, we know a posteriori that $\tilde{\varphi}_i = \varphi_i$. In other words, we are still considering a common atlas on the base curve, and ψ^i is given by gauge-transformations.

Moreover, ψ^i is the unique isomorphism satisfying (17) with respect to our coordinates. Indeed, the first condition implies that such an isomorphism may not depend on the parameter. Thus the second condition provides uniqueness. Recall that ψ^i is given a priori only in a small neighborhood of the initial parameter. Yet the uniqueness of ψ^i implies that ψ^i can be continued to an isomorphism over $\widetilde{\mathcal{U}}^i$ following the analytic continuation of the diffeomorphism on the base curve.

Since $(\widetilde{\mathcal{E}}^*, \widetilde{\nabla}^*)$ and $F^*(\mathcal{E}^*, \nabla^*)$ are two non-singular connections defined on the same base curve and having the same monodromy representation, the Riemann-Hilbert correspondence provides a unique extension ψ^* of the isomorphism ψ_0^*, both given by gauge transformations, such that

$$\begin{cases} \psi^* : (\widetilde{\mathcal{E}}^*, \widetilde{\nabla}^*) \xrightarrow{\sim} F^*(\mathcal{E}^*, \nabla^*) \\ \psi^*|_{\tilde{t}=\tilde{t}_0} = \psi_0^*. \end{cases}$$

We get a commuting diagramm

$$\begin{tikzcd} F^*(\mathcal{E}^*, \nabla^*) \arrow[r, "\Phi^i"] & F^*(\mathcal{E}^i, \nabla^i) & & F^*(\mathcal{E}, \nabla) \\ (\widetilde{\mathcal{E}}^*, \widetilde{\nabla}^*) \arrow[u, "\psi^*"] \arrow[r, "\widetilde{\Phi}^i"'] & (\widetilde{\mathcal{E}}^i, \widetilde{\nabla}^i) \arrow[u, "\psi^i"'] & \rightsquigarrow & (\widetilde{\mathcal{E}}, \widetilde{\nabla}). \arrow[u, "\psi"] \end{tikzcd}$$

inducing a unique isomorphism ψ, given by gauge transformations, such that

$$\begin{cases} \psi : (\widetilde{\mathcal{E}}, \widetilde{\nabla}) \xrightarrow{\sim} F^*(\mathcal{E}, \nabla) \\ \psi|_{\tilde{t}=\tilde{t}_0} = \psi_0. \end{cases}$$

3. The above argumentation has shown that the triple (f, F, Ψ) is unique if (h, H) is unique. Now the classifying map h is always unique, whereas H is unique if, and only if, there is no non-trivial isomorphism of the universal Teichmüller curve fixing (X_0, D_0). This is precisely the case when (g, m) is different from $(0, 0), (0, 1), (0, 2)$ and $(1, 0)$.

4. In the cases $(0, 0), (0, 1), (0, 2)$ and $(1, 0)$ however, the map H may be composed by non-trivial automorphisms of the marked curve.

   In the case $(0, 0)$, the connection (E_0, ∇_0) is the trivial connection on the trivial vector bundle on $\mathbf{P}^1$ and any isomonodromic deformation of this connection is trivial.

   In the cases $(0, 1), (0, 2)$ with $n \geq 3$, it is possible to define a normalized parameter space J, and there is a unique map H such that the image of the map G is contained in this normalized space (see section 3.3).

   In the case $(1, 0)$, it is possible to refer to the case $(1, 1)$ and thus to restore unicity, by fixing a section $\mathcal{T} \to \mathcal{X}_{\mathcal{T}}$ (see section 3.3).

3.3 Special cases with automorphisms

If $3g - 3 + m$ is negative, then the dimension of the Teichmüller space is zero. Moreover, in the special case $g = 1$, $n = 0$, the dimension of the Teichmüller space is one. The parameter space of the isomomonodromic deformation constructed above then is strictly greater than $3g - 3 + n$. On the other hand, there are one-parameter families of automorphisms of the punctured curve exactly in these cases. Depending on the context, it may be of interest to take into account those automorphisms.

Assume now $3g - 3 + n \geq 0$. In this case, we may restore the universal property of the universal isomonodromic deformation.

1. **The case g=0**

In the case of the Riemann sphere we will be able to diminish the dimension of T by means of a quotient in order to get dimension $\sup\{0, 3g-3+n\}$ again. Let us now consider the universal isomonodromic deformation for

$$m-3<0, \text{ but } n-3>0.$$

Consider a tracefree rank 2 connection $(\mathcal{E}, \nabla)$ on the Riemann sphere with no poles except 0 and ∞, with coordinates x in a neigborhood of 0 and $\tilde{x}$ in a neighborhood of ∞, where $\tilde{x}=\frac{1}{x}$.

a) **The case m=1**

Let us consider the case when ∇ has only one pole of multiplicity n. We may suppose this pole is $\{x=0\}$. Now apply the construction of the previous section, but in restriction to the following parameter space of local jets fixing zero:

$$J=\{1\}\times\{0\}\times\mathbf{C}^{n-3}.$$

The group of automorphisms $\mathrm{Aut}(\mathbf{P}^1,0)$ of the marked surface $(\mathbf{P}^1,0)$ acting on our gluing construction is $\{\frac{\lambda x}{1-\mu x} \mid \lambda\in\mathbf{C}^*, \mu\in\mathbf{C}\}$. Such an isomorphism is acting on a jet

$$x+s_3x^3+\ldots+s_{n-1}x^{n-1}$$

in the following way :

$$\lambda x+\lambda\mu x^2+\sum_{i=3}^{n-1}\left(\lambda\mu^{i-1}+\sum_{l=3}^{i}s_l\binom{i-1}{l-1}\lambda^l\mu^{i-l}\right)x^i.$$

With the help of the automorphisms we may thus recover the whole space of $(n-1)-$ jets from the previous section

$$\mathrm{Aut}(\mathbf{P}^1,0)\times J\cong\mathbf{C}^*\times\mathbf{C}^{n-2}$$

and their universal covers will be naturally identified. Yet this new defined isomonodromic deformation will have the universal property.

b) **The case m=2**

Let n_0 (resp. n_∞) be the multiplicity of the poles of the connection ∇ at zero (resp. at infinity), such that $n=n_0+n_\infty$. We assume again $n_0>1$. In this case, we restrict the universal isomonodromic deformation to the universal cover of the set $J^0\times J^\infty$ of local jets, where

$$J^0=\{1\}\times\mathbf{C}^{n_0-2}\quad J^\infty=\mathbf{C}^*\times\mathbf{C}^{n_\infty-2}.$$

The group of automorphisms of the marked surface $\mathbf{P}^1$ fixing zero and infinity is then $\{\lambda x \mid \lambda\in\mathbf{C}^*\}$. Such an isomorphism acts on a pair of jets

$$(x+s_2^0x^2+\ldots+s_{n_0-1}^0x^{n_0-1}, s_1^\infty\tilde{x}+s_2^\infty\tilde{x}^2+\ldots+s_{n_\infty-1}^\infty\tilde{x}^{n_\infty-1})$$

in the following way :

$$(\lambda x + \lambda^2 s_2^0 x^2 + \ldots + \lambda^{n_0-1} s_{n_0-1}^0 x^{n_0-1}, \frac{s_1^\infty}{\lambda}\tilde{x} + \frac{s_2^\infty}{\lambda^2}\tilde{x}^2 + \ldots + \frac{s_{n_\infty-1}^\infty}{\lambda^{n_\infty-1}}\tilde{x}^{n_\infty-1})$$

Again we recover the whole parameter space of the previous section and restore the universal property.

2. **The case g=1, m=0**
Recall that in this case the universal isomonodromic deformation is constructed by suspension. We obtain a non-singular connection on the universal curve having parameter space $\mathbf{H}$. Notice that this connection is invariant under the automorphisms $z \mapsto z + \lambda(\tau)$ of the universal curve

$$\mathbf{H} \times \mathbf{C} \diagup_{\sim},$$

where $(\tau, z) \sim (\tau, z + k_1\tau + k_2)$. As a method to restore the universal property, we may fix a supplementary point on the base curve. Let us fix the zero-section $(\tau, 0)$ of the universal curve for instance.

4 Explicit example

If the initial connection is an irreducible, tracefree rank 2 connection with four poles (counted with multiplicity) over $\mathbf{P}^1$ and the underlying vector bundle is trivial, then its universal isomonodromic deformation implicitely defines a solution $q(t)$ of the Painlevé equation with the associated initial parameters. Indeed, the vector bundle $\mathcal{E}$ underlying the universal isomonodromic deformation is trivial in restriction to a generic parameter t, according to corollary 1.2. The vector bundle $\mathcal{E}$ can be trivialized globally by bimeromorphic gauge transformations, which are in fact holomorphic in restriction to the parameter space $T \setminus \Theta$ (see paragraph 3 in [Mal83a], for example), where the *exceptional set* Θ is a strict closed analytic subset corresponding to the set of parameters t, such that the associated vector bundle E_t is non-trivial, more precisely $E_t = \mathcal{O}(1) \oplus \mathcal{O}(-1)$. Once the vector bundle $\mathcal{E}$ is trivialized, one obtains $q(t)$ directly from the system matrix, up to a normalization. For completeness, we recall that such a solution $q(t)$ may have poles beyond the exceptional set Θ in general. The parameter space on which the Painlevé equations are defined is the Riemann sphere minus the polar set of the initial connection. Yet the solutions of these equations are well defined only on the universal cover of this parameter space. We notice that our construction provides consistent parameter spaces (see [Oka86]).

poles $x_1, .., x_m$	multiplicity $n_1, .., n_m$	parameter space T	Painlevé equation
$0, 1, t, \infty$	$1, 1, 1, 1$	$\widetilde{\mathbf{P}^1 \setminus \{0, 1, \infty\}} = \mathbf{H}$	P_{VI}
$0, 1, \infty$	$2, 1, 1$	$\widetilde{\mathbf{C}^*}$	P_{V}
$0, \infty$	$3, 1$	$\mathbf{C}$	P_{IV}
$0, \infty$	$2, 2$	$\widetilde{\mathbf{C}^*}$	P_{III}
0	4	$\mathbf{C}$	P_{II}

Let us construct an example of an isomonodromic deformation of an irreducible tracefree rank 2 connection with four simple poles over $\mathbf{P}^1$, which thus will correspond to a solution of a Painlevé VI equation, where we describe explicitely the exceptional set Θ.

Example 4.1. *Consider the connection on the trivial bundle over the* $\mathbf{P}^1$ *given in a neighborhood of the origin by the system*

$$dY = \begin{pmatrix} -\frac{\theta}{2x} & 0 \\ 0 & \frac{\theta}{2x} \end{pmatrix} Y dx, \tag{18}$$

with $\theta \in \mathbf{C} \setminus \{0,1\}$. *In order to get a fuchsian system with four poles in* $0, 1, t$ *and* ∞, *we apply two elementary transformations centered in* $(1,(1,1))$ *and* $(t,(1,s))$ *in* $\mathbf{P}^1 \times \mathbf{C}^2$ *and we renormalize in order to get a tracefree connection again :*

$$\widetilde{Y} = \frac{1}{\sqrt{(x-1)(x-t)(1-s)}} \begin{pmatrix} x-1 & 0 \\ 0 & x-t \end{pmatrix} \begin{pmatrix} s & -1 \\ 1 & -1 \end{pmatrix} Y.$$

We thereby define an isomonodromic deformation parametrized by the universal cover $\pi : \mathbf{H} \to \mathbf{P}^1 \setminus \{0,1,\infty\}$ *if, and only if,* $(t, s(t))$ *is a leaf of the Riccati foliation* $dy = \frac{\theta}{x} y dx$ *associated to (18). Thus we have to ask* $s(t) = ct^\theta$ *for some constant* c. *Now the self-intersection number of a section* σ *of the associated* $\mathbf{P}^1$*-bundle is shifted by* -1 *(resp.* $+1$*) by an elementary transformation centered on a point lying on* σ *(resp. not lying) on* σ. *Thus the minimal self-intersection number remains zero if, and only if,* s *is different from* 1. *Otherwise the minimal self-intersection number becomes* -2. *In other words, the degree of stability of the vector bundle* $E_{\tilde{t}}$ *with* $\tilde{t} \in \mathbf{H}$ *along this isomonodromic deformation is*

$$\kappa(E_{\tilde{t}}) = \begin{cases} -2 & \text{if } t \in \{\exp((n-\tilde{c})\frac{2i\pi}{\theta}),\ n \in \mathbf{Z}\} \\ 0 & \text{else} \end{cases}$$

where $c = \exp(\tilde{c}2i\pi)$, *and* $t = \pi(\tilde{t})$. *If we calculate the associated solution of the Painlevé*

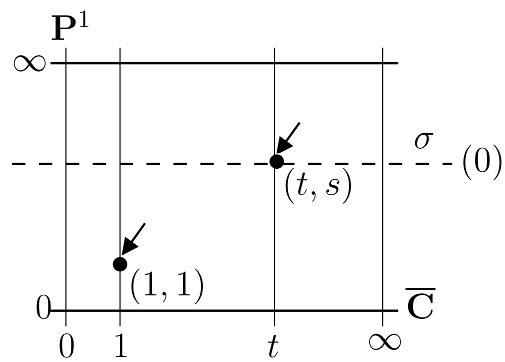

Figure 5: Construction of an isomonodromic deformation by elementary transformations with parameter

VI equation (see [Lor07]), we have to normalize the system such that the residue matrix A_∞ *at infinity is diagonal.*

$$\widetilde{\widetilde{Y}} = \frac{1}{\sqrt{1-s}} \begin{pmatrix} -1 & 1 \\ -1 & s \end{pmatrix} \widetilde{Y}.$$

By these birational gauge-transformations we get a system $\mathrm{d}\tilde{\tilde{Y}} = A\tilde{\tilde{Y}}$ *where the* $\mathrm{d}x$*-part of the matrix* A *is the following:*

$$\frac{\theta}{2x}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathrm{d}x+$$

$$\frac{\frac{(1-t)}{(1-s)^2}}{x(x-1)(x-t)}\begin{pmatrix} -\frac{1}{2}(1-s^2)x+\theta s(1-t) & -(\theta-1)(1-s)x+\theta(t-s) \\ -(\theta+1)s(1-s)x+\theta s(1-st) & \frac{1}{2}(1-s^2)x-\theta s(1-t) \end{pmatrix}\mathrm{d}x$$

The zero of the $(1,2)$*-coefficient of* A *then is the solution*

$$q(t) = \frac{\theta}{\theta-1}\frac{t-s}{1-s}$$

of the associated Painlevé VI equation :

$$\begin{aligned} \frac{\mathrm{d}^2q}{\mathrm{d}t^2} &= \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{\mathrm{d}q}{\mathrm{d}t} \\ &\quad +\frac{q(q-1)(q-t)}{2t^2(t-1)^2}\left(\kappa_\infty^2-\kappa_0^2\frac{t}{q^2}+\kappa_1^2\frac{t-1}{(q-1)^2}+(1-\kappa_t^2)\frac{t(t-1)}{(q-t)^2}\right) \end{aligned}$$

with coefficients

$$(\kappa_0,\kappa_1,\kappa_t,\kappa_\infty) = (\theta,1,1,\theta-1).$$

5 Proof of the main result

Let $(P_t \to X_t, \mathcal{F}_t)_{t\in T}$ be the projectivization of the universal isomonodromic deformation of some irreducible tracefree rank 2 connection with n (resp. m) poles counted with (resp. without) multiplicity, on a compact Riemann surface of genus g. We will assume

$$\dim(T) = 3g-3+n$$

and $3g-3+n > 0$. In particular we exclude the special cases $(g,n) = (0,0),(0,1)$, $(0,2)$, $(0,3)$ and $(1,0)$, that will be treated in section 6.2. Moreover, in the cases $(g,m) = (0,1),(0,2),(0,3)$ we use the universal isomonodromic deformation with normalized parameter space, as in section 3.3.

We are now going to prove the main theorem. Let k be an integer. We denote by T_k the following subset of the parameter space :

$$T_k = \{t \in T \mid \exists \text{ section } \sigma_t \text{ of } P_t \text{ such that } \sigma_t\cdot\sigma_t \leq k\}.$$

We are going to prove that

$$\mathrm{codim}(T_k) \geq g-1-k.$$

Recall that we have

$$T_g = T$$

according to a result of M. Nagata. It follows from $\mathrm{codim}(T_{g-2}) \geq 1$ that if t is generic, then P_t has minimal self-intersection number g or $g-1$, according to the parity of g.

5.1 Filtration of the parameter space

Lemma 5.1 (Semi-continuity)**.** *For each integer k, the set T_k is a closed analytic subset of T. In particular, we have an increasing filtration by closed analytic sets*

$$\ldots \subset T_{g-3} \subset T_{g-2} \subset T_{g-1} \subset T_g = T.$$

Outline of the proof: It is sufficient to consider the *germified* parameter space (T, t_0) for some initial parameter t_0. According to a theorem of M. Hakim [Hak72], the analytic family $(E_t)_{t\in T}$ of holomorphic vector bundles is analytically equivalent to an analytic family $(E_t^{\mathrm{alg}})_{t\in T}$ of algebraic vector bundles. In particular, the vector bundle $\mathcal{E}$ is generated by a finite number of global meromorphic sections. Up to small perturbations, each such section defines a holomorphic section of the projective bundle $\mathbf{P}(\mathcal{E})$. We may choose three such sections σ_0, σ_1 and σ_∞. Again by small perturbations we make sure that these sections are in general position:

$$\begin{cases} \sigma_0, \sigma_1, \sigma_\infty \text{ are pairwise transverse} \\ \sigma_0 \cap \sigma_1 \cap \sigma_\infty = \emptyset \end{cases}$$

After applying elementary transformations centered in each of the intersections $p_1, \ldots, p_N$, we get the trivial $\mathbf{P}^1$-bundle over $\mathcal{X}$. Note that the position of the centers $q_1, \ldots, q_N$ for the inverse elementary transformations on $\mathcal{X} \times \mathbf{P}^1$ is depending holomorphically on the parameter $t \in T$. For some fixed parameter t_1, let σ_{t_1} be a section of $\mathcal{P}_{t_1}$ and let $\tilde{\sigma}_{t_1}$ the

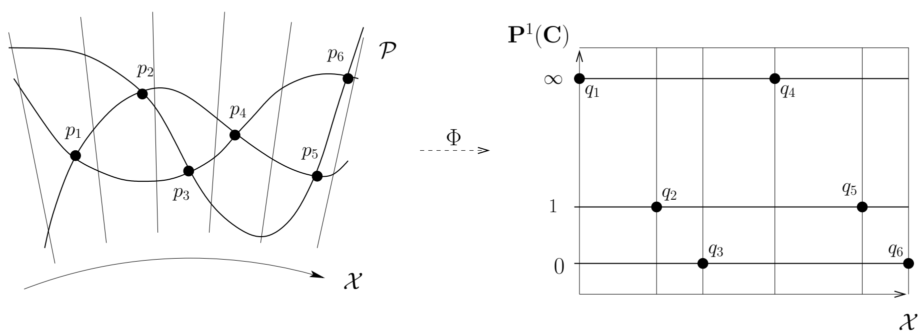

Figure 6: We can obtain the trivial bundle by elementary transformations in the intersection points.

section of $\mathcal{X} \times \mathbf{P}^1|_{t=t_1}$ resulting from the elementary transformations. Let d be the degree of the section $\tilde{\sigma}_{t_1}$ on the trivial bundle. Then the self-intersection number of σ_{t_1} is

$$\sigma_{t_1} \cdot \sigma_{t_1} = 2d + N - 2\eta, \tag{19}$$

where η is the number of points lying on $\tilde{\sigma}_{t_1}$ within $q_1(t_1), \ldots, q_N(t_1)$. Thus there is a section of self-intersection number loss or equal to k of the bundle P_{t_1} if, and only if, there is a pair (d, η) satisfying

$$k \geq 2d + N - \eta$$

and a selection of η points within $q_1(t_1), \ldots, q_N(t_1)$ such that there is an irreducible bidegree $(1, d)$-curve (the graph of a degree d rational function) passing by these points. Thus the existence of a section with self-intersection number less or equal to k is an algebraic condition on the position of the points $q_1(t_1), \dots, q_N(t_1)$. In varying the parameter, we see that the set T_k is the pull-back of an algebraic set by an analytic mapping. This means T_k is a closed analytic subset of T. □

Lemma 5.2. *Let k be an integer and suppose let t_k be a generic parameter in $T_k \setminus T_{k-1}$. Then in restriction to the germified parameter space (T_k, t_k), there is a holomorphic section Σ of $\mathcal{P}$ inducing for each parameter t in the germ (T_k, t_k) a holomorphic section $\sigma_t = \Sigma|_t$ of P_t satisfying*

$$\sigma_t \cdot \sigma_t = k.$$

In particular, Σ is isomorphic to the germ of the universal curve $\mathcal{X}$ over (T_k, t_k).

Outline of the proof: As in the proof of the previous lemma we can see the germ T_k as a finite union over possible configurations of closed analytic subsets parametrizing families of generically holomorphic sections. Choose one such subset which is equal to the germ T_k. At a generic parameter t_k in $T_k \setminus T_{k-1}$, we can suppose that the induced section is holomorphic. □

5.2 A formula of M. Brunella

Let k be an integer and $t_k \in T_k \setminus T_{k-1}$. Let p be a point of σ_{t_k} In a neighborhood of p, the foliation $\mathcal{F}_{t_k}$ is given by a meromorphic 1-form

$$\omega_{t_k} : \ \mathrm{d}y + \left(\frac{1}{x^l} a(x) y^2 + \frac{1}{x^l} b(x) y + \frac{1}{x^l} c(x) \right) \mathrm{d}x,$$

where $l \geq 0$ is the order of the pole of $\mathcal{F}_{t_k}$ at $\{x = 0\}$, and the section σ_{t_k} is given by a reduced local equation $\{f = 0\}$. Alternatively, the foliation $\mathcal{F}_{t_k}$ can be defined in our coordinates by the holomorphic vector field

$$\mathcal{V} : \ x^l \frac{\partial}{\partial x} - (a(x)y^2 + b(x)y + c(x)) \frac{\partial}{\partial y}.$$

As in [Bru04], page 22, the *multiplicity of tangency* between $\mathcal{F}_{t_k}$ and σ_{t_k} at p then is given by

$$\mathrm{tang}_p(\mathcal{F}_{t_k}, \sigma_{t_k}) = \frac{\mathcal{O}_p}{< f, \mathcal{V}(f) >},$$

where $\mathcal{O}_p$ is the local algebra of P_{t_k} at p and $< f, \mathcal{V}(f) >$ is the ideal generated by f and its Lie-derivative $\mathcal{V}(f)$ along $\mathcal{V}$. Now the fact that the initial connection is irreducible implies that no section of the projective bundle, and in particular not σ_{t_k}, can be invariant for the associated Riccati foliation. This means that $\mathcal{V}(f)$ is not identically zero.

Example 5.3. *For example, let us choose local coordinates (x, y) of P_{t_k}, such that the section σ_{t_k} is given by $\{y = \infty\}$. Then for $p = (0, \infty)$, we get*

$$\#\mathrm{tang}_{(0,\infty)}(\mathcal{F}_{t_k}, \sigma_{t_k}) = \mathrm{ord}_0(a(x)).$$

The fact that σ_{t_k} is not invariant by $\mathcal{F}_{t_k}$ then means that $a(x)$ is not identically zero.

In [Bru04], page 23, M. Brunella stated that the total number of tangencies on σ_{t_k} is related to the self-intersection number of σ_{t_k} in the following way :

$$\#\mathrm{tang}(\mathcal{F}_{t_k}, \sigma_{t_k}) = \sigma_{t_k} \cdot \sigma_{t_k} - T_{\mathcal{F}_{t_k}} \cdot \sigma_{t_k}, \tag{20}$$

where $T_{\mathcal{F}_{t_k}}$ is the tangent bundle of $\mathcal{F}_{t_k}$. Let us recall Brunella's proof.

Proof: There is a covering by local trivialization charts $(U_i \times \mathbf{P}^1)$ of P_{t_k} such that $\mathcal{F}_{t_k}$ and σ_{t_k} are given on $U_i \times \mathbf{P}^1$ respectively by a holomorphic vector field $\mathcal{V}_i$ and a reduced holomorphic equation $\{f_i = 0\}$. They are gluing together by means of holomorphic transition maps $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ respectively $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. We have

$$\begin{aligned} \mathcal{V}_i &= g_{ij} \mathcal{V}_j \\ f_i &= f_{ij} f_j. \end{aligned}$$

Here the cocyle (g_{ij}) is defining the cotangent bundle $T^*_{\mathcal{F}_{t_k}}$, which is the dual of the tangent bundle $T_{\mathcal{F}_{t_k}}$ by definition. By Leibniz's rule, on $(\mathcal{O}(U_i \cap U_j)) \times \mathbf{P}^1$ we have

$$\mathcal{V}_i(f_i) = g_{ij}(f_{ij} \mathcal{V}_j(f_j) + f_j \mathcal{V}_j(f_{ij})).$$

Since $f_j = 0$ on σ_{t_k}, this implies that $(\mathcal{V}_i(f_i))|_{\sigma_{t_k}}$ is defining a global holomorphic section of the bundle

$$[T^*_{\mathcal{F}_{t_k}} \otimes \mathcal{O}_{P_{t_k}}(\sigma_{t_k})]|_{\sigma_{t_k}}.$$

The zeroes of this section (counted with multiplicity) are exactly the tangencies between $\mathcal{F}_{t_k}$ and σ_{t_k}. Thus

$$\begin{aligned} \#\mathrm{tang}(\mathcal{F}_{t_k}, \sigma_{t_k}) &= \deg([T^*_{\mathcal{F}_{t_k}} \otimes \mathcal{O}_{P_{t_k}}(\sigma_{t_k})]|_{\sigma_{t_k}}) \\ &= \deg(T^*_{\mathcal{F}_{t_k}}|_{\sigma_{t_k}}) + \deg(\mathcal{O}_{P_{t_k}}(\sigma_{t_k})|_{\sigma_{t_k}}) \\ &= T^*_{\mathcal{F}_{t_k}} \cdot \sigma_{t_k} + \sigma_{t_k} \cdot \sigma_{t_k} \\ &= \sigma_{t_k} \cdot \sigma_{t_k} - T_{\mathcal{F}_{t_k}} \cdot \sigma_{t_k}. \end{aligned}$$

□

Lemma 5.4. *Since $\mathcal{F}_{t_k}$ is a Riccati foliation, formula (20) implies*

$$\#\mathrm{tang}(\mathcal{F}_{t_k}, \sigma_{t_k}) = \sigma_{t_k} \cdot \sigma_{t_k} + 2g - 2 + n, \tag{21}$$

where g is the genus of the Riemann surface X_{t_k} and n is the number of vertical leaves of $\mathcal{F}_{t_k}$, counted with multiplicity.

Proof: Let $x_1, \ldots, x_m$ on X_{t_k} be the poles of $\mathcal{F}_{t_k}$ with multiplicity $n_1, \ldots, n_m$ respectively. Denote by $\pi : P_{t_k} \to X_{t_k}$ the projection of the $\mathbf{P}^1$-bundle to the base curve. Let v_0 be a meromorphic vector field on X_{t_k}, *i.e.* a meromorphic section of the sheaf of holomorphic vector fields on X_{t_k}. Its divisor $K^* = (v_0)_0 - (v_0)_\infty$ thus is dual to the canonical divisor K, and we have

$$\deg(K^*) = 2 - 2g.$$

We know that v_0 lifts in a unique way to a vector field $\mathcal{V}_0$ tangent to the foliation with divisor

$$\pi^* K^* - \sum_{i=1}^{m} n_i [\pi^* x_i],$$

which is topologically equivalent to $-(2g-2+n)f$, where f is a generic fibre of P_{t_k}. Since σ_{t_k} is a section, we thus have $T_{\mathcal{F}_{t_k}} \cdot \sigma_{t_k} = -(2g-2+n)$ □

In particular, we have

$$\#\mathrm{tang}(\mathcal{F}_{t_k}, \sigma_{t_k}) = k + 2g - 2 + n. \tag{22}$$

Note that these tangencies occur either in a non-singular point of the foliation by a contact with the section or in a singular point of the foliation lying on the section. Otherwise there is transversality.

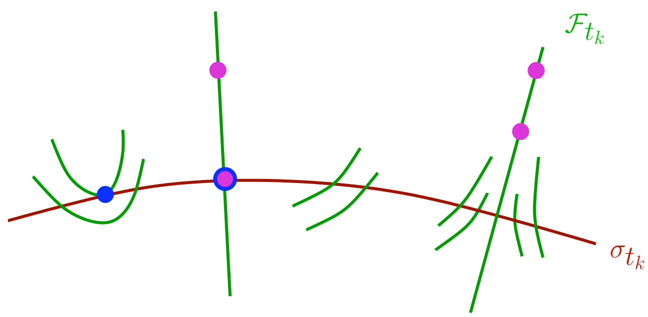

Figure 7: Different possibilities of tangency respectively transversality

5.3 Trivial deformations for small self-intersection numbers

In order to prove theorem 1.1, it is sufficient to consider integers k such that

$$T_k \setminus T_{k-1} \neq \emptyset,$$

i.e. T_k is a stratum in lemma 5.1 such that k is the generic minimal self-intersection number among the ruled surfaces P_t with parameter $t \in T_k$. Let t_k be a generic parameter in T_k. From now on we consider the projectivized universal isomonodromic deformation in restriction to the germified parameter space (T_k, t_k). In order to simplify notations, we will denote the resulting isomonodromic deformation by $(\mathcal{P} \to \mathcal{X}, \mathcal{F})$ as well. Since t_k is generic, we will suppose that the minimal self-intersection number as well as the number of tangencies is constant along the isomonodromic deformation $(P_t \to X_t, \mathcal{F}_t)_{t \in T_k}$. Let Σ be a holomorphic section of $\mathcal{P}$ over $\mathcal{X} \to (T_k, t_k)$ inducing for each parameter $t \in (T_k, t_k)$ a holomorphic section of P_t with self-intersection number k, as in lemma 5.2. We need to show $\dim(T_k) \leq k + 2g - 2 + n$. Since $\#\mathrm{tang}(\mathcal{F}_{t_k}, \sigma_{t_k}) = k + 2g - 2 + n$, we actually want to show

$$\dim(T_k) \leq \#\mathrm{tang}(\mathcal{F}_{t_k}, \sigma_{t_k}). \tag{23}$$

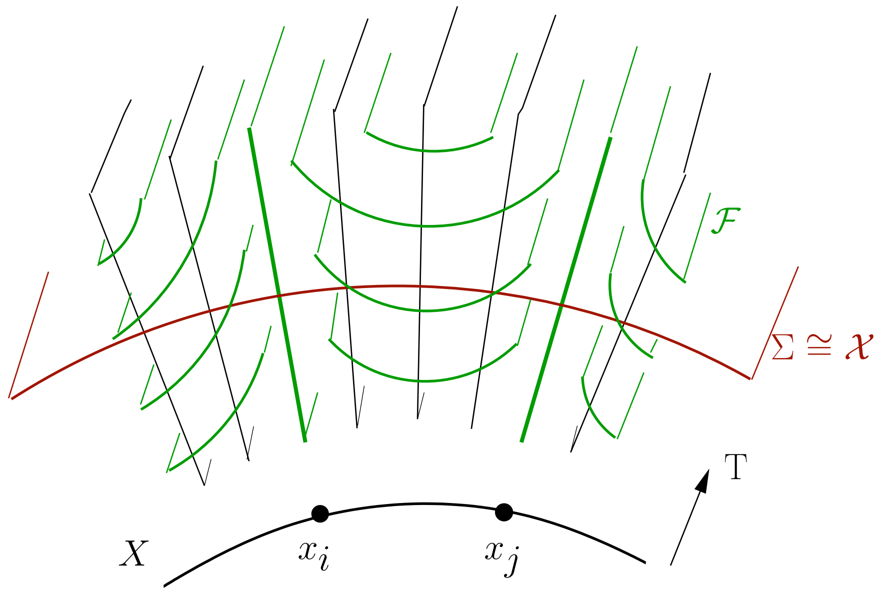

Figure 8: Deformation of a Riccati foliation and global section

We will see that if there are no tangencies, then the projectivized universal isomonodromic deformation $(\mathcal{P} \to \mathcal{X}, \mathcal{F})$ is trivial in restriction to (T_k, t_k). In the general case, we will show that the projectivized universal ismonodromic deformation is trivial in restriction to a submanifold $T' \subset T_k$ of dimension

$$\dim(T') \geq \dim(T_k) - \#\mathrm{tang}(\mathcal{F}_{t_k}, \sigma_{t_k}). \tag{24}$$

Lemma 5.5. *If $(\mathcal{P} \to \mathcal{X}, \mathcal{F})|_{T'}$ with parameter space (T', t') is trivial, then $(\mathcal{E} \to \mathcal{X}, \nabla)|_{T'}$ is trivial, too. In particular, this implies $\dim(T') = 0$ according to the universal poperty theorem.*

Proof: The foliation $(\mathcal{P} \to \mathcal{X}, \mathcal{F})|_{(T',t')}$ is trivial, *i.e.* isomorphic to the constant foliation $(P_{t'} \to X_{t'}, \mathcal{F}_{t'}) \times T'$ by gauge-coordinate transformation, if, and only if, there is a vector field v on $\mathcal{X}|_{T'}$ transverse to the parameter, that lifts to a vector field V on $\mathcal{P}|_{T_k}$ tangent to the foliation $\mathcal{F}|_{T_k}$. In each non-singular point however, the vector field v lifts on $\mathcal{E}|_{T'}$ to a vector field $\widetilde{V}$ tangent to the flat connection $\nabla|_{T'}$. The vector field $\widetilde{V}$ can be continued analytically at the polar set. Indeed, in a neighborhood of a pole, choose coordinates (t, x, y) on $\mathcal{P}|_{T'}$, such that the vector fields v and V are given by $\frac{\partial}{\partial t}$. In these coordinates, $\mathcal{F}|_{T'}$ does not depend on the parameter t :

$$\mathcal{F}|_{T'} : x^l \mathrm{d}y = (\alpha(x)y^2 + \beta(x)y + \gamma(x))\mathrm{d}x. \tag{25}$$

Since $(\mathcal{E} \to \mathcal{X}, \nabla)|_{(T',t')}$ is a tracefree connection, its connection matrix in the coordinates (t, x, Y) is uniquely determined by (25) :

$$\nabla|_{T'} : x^l \mathrm{d}Y = \begin{pmatrix} -\frac{\beta(x)}{2} & -\alpha(x) \\ \gamma(x) & \frac{\beta(x)}{2} \end{pmatrix} Y \mathrm{d}x.$$

The vector field v on the base curve thus lifts to the vector field $\widetilde{V} = \frac{\partial}{\partial t}$ tangent to the connection. $\square$

If the projectivised universal isomonodromic deformation is trivial in restriction to T', then we have $\dim(T') = 0$, and (23) follows.

5.3.1 Proof in the logarithmic or non-singular case : triviality of the curve deformation

In the (non-singular or) logarithmic case, each isomonodromic deformation fixing the curve and the poles is trivial according to the Riemann-Hilbert correspondence. In this case it is sufficient to find a submanifold T' of T_k of dimension (24), such that

$$(\mathcal{X}|_{T'}, \mathcal{D}|_{T'}) \cong (T' \times X_{t_k}, T' \times D_{t_k}). \tag{26}$$

a) **Case of transversality**
Since the number of tangencies given by $k+2g-2+n$ is a positive integer, we already know that

$$k \geq 2 - 2g - n.$$

As an example, let us now consider the case $k = 2-2g-n$. By (22), the foliation $\mathcal{F}_{t_k}$ associated to the generic initial parameter t_k is transverse to the section σ_{t_k} in this case. Furthermore, the foliation $\mathcal{F}$ is transverse to the parameter $\{t = t_k\}$ by local constancy. Hence the induced foliation $\mathcal{F}|_\Sigma$ is a non-singular foliation of codimension 1, which is also transverse to the parameter $\{t = t_k\}$. We may conclude that the

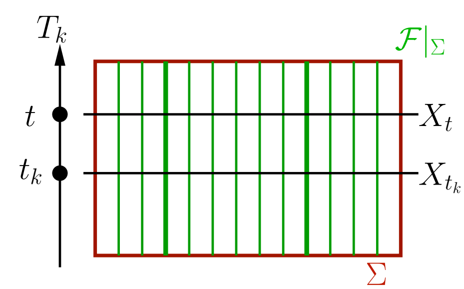

Figure 9: Case of transversality: the induced foliation trivializes the fibration by curves

induced foliation trivializes the fibration by curves $(X_t)_{t\in T_k}$ over the germ T_k. More precisely, we may choose local coordinates such that $\mathcal{F}$ and Σ are given respectively by

$$\mathcal{F}: \ \mathrm{d}y + \alpha(t,x)y^2 + \beta(t,x)y + \gamma(t,x), \quad \Sigma : \{y = \infty\}.$$

Then the induced foliation $\mathcal{F}|_\Sigma$ is given by

$$-\alpha(t,x).$$

Write $-\alpha$ in the form

$$-\alpha = \frac{1}{x^l}(u(t,x)\mathrm{d}x + v(t,x)\mathrm{d}t),$$

where $l \geq 0$ is the order of the pole of $\mathcal{F}$ in these coordinates. Since $\mathcal{F}_{t_k}$ is transversal to σ_{t_k}, we have $u(t_k, 0) \neq 0$. If $l = 0$, this implies that there is a coordinate change $(t, x) \mapsto (t, \varphi(t, x))$ such that $-\alpha$ has the form $\tilde{u}(t, x)\mathrm{d}x$ with $\tilde{u}(t_k, 0) \neq 0$. In a small neighborhoof of $(t_k, 0)$, the foliation $\mathcal{F}|_\Sigma$ then is given by

$$\mathcal{F}|_\Sigma : \ \mathrm{d}x.$$

If $l > 0$, then the transversality condition on $\mathcal{F}$ implies that v is of the form $v(t, x) = xw(t, x)$. This implies there is a coordinate change $(t, x) \mapsto (t, \varphi(t, x))$ fixing $\{x = 0\}$ such that $-\alpha$ has the form $\frac{1}{x^l}\tilde{u}(t, x)\mathrm{d}x$ with $\tilde{u}(t_k, 0) \neq 0$. In a small neighborhoof of $(t_k, 0)$, the foliation $\mathcal{F}|_\Sigma$ then is given by

$$\mathcal{F}|_\Sigma : \ \frac{1}{x^l}\mathrm{d}x.$$

Up to appropriate coordinate transformations, the reduced version of the foliation $\mathcal{F}|_\Sigma$ thus is given in each chart with coordinates (t, x) by $\mathrm{d}x = 0$. The base curve $\mathcal{X}$ then is trivial in these coordinates, *i.e.* it is given by locally trivial charts $T_k \times U$ with coordinates (t, x) such that the transition maps do not depend on the parameter. Otherwise a $\mathrm{d}t$-component would appear for $\mathcal{F}|_\Sigma$. Moreover, since the position of the poles on Σ is indicated by special leaves of $\mathcal{F}|_\Sigma$, the polar divisor does not depend on the parameter either. We thus have trivialized simultaneously the curve and the polar divisor. On the non-singular or logarithmic case (*i.e.* if the order l of each pole is 0 or 1), the universal isomonodromic deformation restricted to the parameter space T_k then is trivial. We thus have (26) as desired, with $T' = T_k$.

b) **Case of tangency**

Let us now consider the general logarithmic case. We are going to construct a submanifold $T' \subset T_k$ of dimension (24) such that the foliation $(\mathcal{F}|_\Sigma)|_{T'}$ is transverse to the parameter $t \in T'$. This will trivialize the fibration by punctured curves along T' and provide (26), as desired.

Since the foliation $\mathcal{F}$ is already transverse to the parameter, each tangency between $\mathcal{F}|_\Sigma$ and the parameter $\{t = t_k\}$ is induced by a tangency between $\mathcal{F}$ and Σ.
Let us consider a local chart $T_k \times U \times \mathbf{P}^1$ of $\mathcal{P}$ containing such a tangency. For appropriate coordinates (t, x, y) we may suppose that Σ is given by $\{y = \infty\}$ in this chart and the tangency between $\mathcal{F}$ and Σ is located in $\{x = 0\}$. Let $\widetilde{\omega}$ be a holomorphic 1-form defining $\mathcal{F}$. We have

$$\widetilde{\omega} = x^l\mathrm{d}y + y^2\tilde{\alpha}(t, x) + y\tilde{\beta}(t, x) + \tilde{\gamma}(t, x), \tag{27}$$

where $\tilde{\alpha}, \tilde{\beta}$ et $\tilde{\gamma}$ are holomorphic 1-forms and $l \in \{0, 1\}$. If ν is the multiplicity of the tangency between $\mathcal{F}|_{t=t_k}$ and σ_{t_k} for a generic parameter t_k, then the foliation $\mathcal{F}|_\Sigma$ given by $-\tilde{\alpha}$ is of the form

$$-\tilde{\alpha} = x^\nu u(t, x)\mathrm{d}x + \sum_{i=0}^{\nu} x^i\omega_i(t) + x^{\nu+1}\omega_{\nu+1}(t, x),$$

with $u|_{\{x=0\}} \not\equiv 0$, where u is a holomorphic function and ω_i are holmorphic 1-forms.

– If $\omega_0 \equiv \ldots \equiv \omega_\nu \equiv 0$, we get $-\tilde{\alpha} = x^\nu u(t,x)\mathrm{d}x + x^{\nu+1}\omega_{\nu+1}(t,x)$ and we may consider the reduced 1-form $-\tilde{\tilde{\alpha}} = u(t,x)\mathrm{d}x + x\omega_{\nu+1}(t,x)$ which defines a non-singular foliation on Σ of codimension 1 generically transverse to the parameter $\{t = t_k\}$ as in a).

  Geometrically, this means that the tangency between $\mathcal{F}_t$ and σ_t remains in the same leaf when t varies in T_k. Then the foliation on Σ induced by $\mathcal{F}$ has one multiple leaf, but the reduced foliation is non-singular and transverse to the parameter $\{t = t_k\}$.

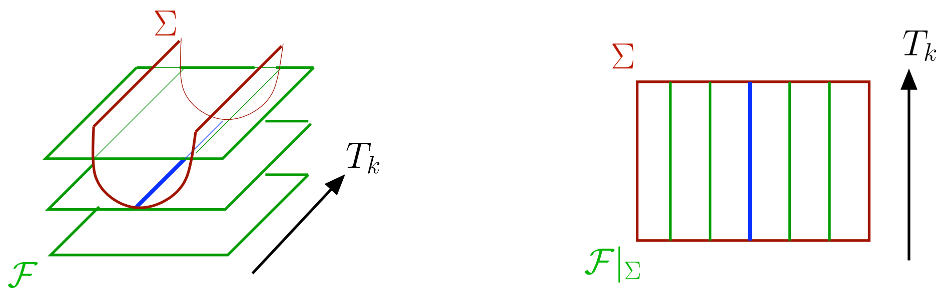

Figure 10: Tangency remains in the leaf: resort to the transversal case

Remark 5.6. *Note that in this case, we do not need to restrict the parameter space.*

– Now consider the case when the tangency does not remain in the same leaf. Then the foliation induced on Σ will not be transverse to the parameter. Using

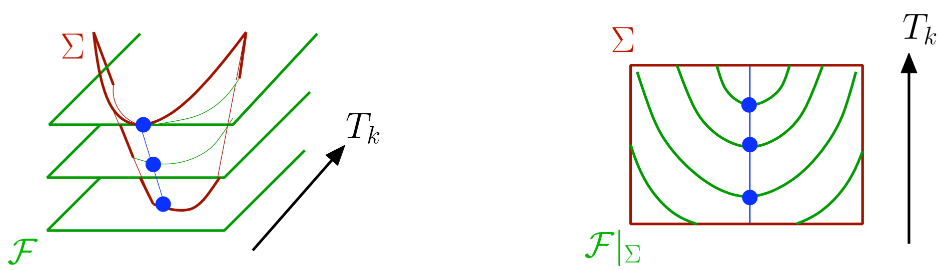

Figure 11: Tangency changes the leaf: we can not conclude

the fact that the leaves of $\mathcal{F}$ have codimension 1, we will find a submanifold T' of codimension 1 of T_k such that the tangency in consideration between $\mathcal{F}_t$ and σ_t remains in the same leaf when t varies in T'.

Let us calculate the integrability condition $0 = \tilde{\alpha} \wedge \mathrm{d}\tilde{\alpha}$ for the foliation $\mathcal{F}|_\Sigma$ explicitely:

$$0 = \sum_{i=0}^{\nu-1} x^i \left[\sum_{j=0}^{i} \omega_{i-j} \wedge \mathrm{d}\omega_j - \left(\sum_{j=0}^{i} (j+1)\omega_{i-j} \wedge \omega_{j+1} \right) \wedge \mathrm{d}x \right] + x^\nu [\ldots] \quad (28)$$

Since ω_i does not depend on x for $i \in \{0, \dots \nu\}$, we get

$$0 = \sum_{j=0}^{i} (j+1)\omega_{i-j} \wedge \omega_{j+1}$$

for each $i \in \{0, \dots, \nu - 1\}$. For $i = 0$ we obtain $\omega_0 \wedge \omega_1 = 0$ and by induction we conclude that all 1-forms ω_i for $i \in \{0, \dots, \nu\}$ are dependent. Note that the integrability condition (28) implies also $\omega_0 \wedge \mathrm{d}\omega_0 = 0$. We may suppose that $\omega_0 \not\equiv 0$, otherwise we consider the reduced version of $-\tilde{\alpha}$. Let T' be one leaf of the possibly singular foliation on T_k defined by $\omega_0 = 0$. Then all the ω_i for $i \in \{0, \dots, \nu\}$ are zero on T'. Since t_k is generic, we may suppose that T' is a smooth subvariety of codimension 1 of the germ T_k.

We proceed in the same way successively with every tangency and we finally get a subspace T' of T_k of codimension less or equal to the number of tangencies counted *without multiplicity*. In particular, we have (24). According to our construction of T', the foliation $(\mathcal{F}_t)_{t \in T'}$ on $(P_t)_{t \in T'}$ induces a foliation on $(\sigma_t)_{t \in T'}$ which is transverse to the parameter $\{t = t_k\}$ and which has only apparent singularities being actually multiple leaves. We may conclude as in the case of transversality and obtain (26).

Remark 5.7. *We remark that in the logarithmic case, all the tangencies between $\mathcal{F}_t$ and σ_t for a generic parameter $t \in T_k$ have to be simple and cannot remain in the same leaf.*

5.3.2 Proof in the case of multiple poles : triviality of the deformation

In the case of multiple poles we will have not only to trivialize the deformation of the curve and the position of the poles but also the deformation of the foliation.

a) **Tangencies in non-singular points of the foliation**
First consider the case when none of the tangencies $\mathrm{tang}(\mathcal{F}_{t_k}, \sigma_{t_k})$ occurs in a singularity of the foliation $\mathcal{F}_{t_k}$ which is lying on the section σ_{t_k}. We may suppose that this is the case for every parameter t in the germified parameter space (T_k, t_k). We can use the method applied in the logarithmic case to trivialize the foliation $\mathcal{F}|_\Sigma$ and thereby the punctured curve $\mathcal{X}$ in restricting the parameter space T_k to a submanifold T' of dimension (24). As before, we choose a new atlas with local charts $T' \times U \times \mathbf{P}^1$ and coordinates (t, x, y) such that in each chart, the induced foliation $\mathcal{F}|_\Sigma$ is given by $\mathrm{d}x = 0$.

Now we want to trivialize locally the foliation $\mathcal{F}|_{T'}$ by gauge transformations. In particular, we will not allow any non-trivial coordinate transformations in (t, x) any more.

- If U is a neighborhood of a non-singular point, then there is a holomorphic gauge transformation such that $\mathcal{F}$ is given by the form $\mathrm{d}y = 0$ on the local chart. After this gauge transformation, the section Σ will be given by $\{y = f(t,x)\}$, where f is a meromorphic function. Yet the induced foliation on Σ will not depend on t. Hence f does not depend on t.

$$\mathcal{F} : \mathrm{d}y = 0, \quad \Sigma : \{y = f(x)\}, \quad \mathcal{F}|_\Sigma : x^\nu \mathrm{d}x = 0$$

– Let U be a chart on $\mathcal{X}$ such that there are no tangencies between $\mathcal{F}$ and Σ above U. Let $\mathcal{F}$ be defined by a meromorphic 1-form

$$\omega = \mathrm{d}y + y^2\alpha(t,x) + y\beta(t,x) + \gamma(t,x) \tag{29}$$

with a pole of order $l \geq 1$ at $\{x = 0\}$. Let $\tilde{\alpha} = \alpha x^l$. We have $\mathcal{F}|_\Sigma : \mathrm{d}x = 0$ and since there is no tangency, $\tilde{\alpha} = a(t,x)\mathrm{d}x$, respectively

$$\alpha = \frac{a(t,x)}{x^l}\mathrm{d}x,$$

where $a(0,t) \neq 0$. Since t_k is generic, we can find a gauge transformation fixing infinity such that $\mathcal{F}$ is given by a normal form

$$\mathcal{F} : \mathrm{d}y + \frac{y^2}{x^l}\mathrm{d}x + \frac{c(x)}{x^l}\mathrm{d}x, \quad \Sigma : \{y = \infty\}, \quad \mathcal{F}|_\Sigma : \mathrm{d}x = 0,$$

as in the proof of lemma 2.9.

We have thus locally trivialized $\mathcal{F}$ and $\mathcal{F}|_\Sigma$ simultaneously over (T', t_k). Consider now a transition map between such charts, given by a gauge-coordinate transformation

$$(t,x,y) \mapsto (t, \varphi(t,x), \phi(t,x) \cdot y).$$

Since Σ and $\mathcal{F}|_\Sigma$ do not depend on the parameter t in our charts, neither does the underlying coordinate transformation:

$$\varphi = \varphi(x).$$

Moreover, since $\mathcal{F}$ does not depend on the parameter t in each local chart, neither does the underlying gauge transformation. Otherwise a non-trivial $\mathrm{d}t$-component would appear in the local 1-form defining $\mathcal{F}$. We have

$$\phi = \phi(x).$$

b) **Example for the limits of this method in the case of tangencies in singular points**
We can try to apply the previous method to a general universal isomonodromic deformation: First we trivialize the deformation of the curve and afterwards we trivialize the foliation $\mathcal{F}$ over the set of non-singular points or points of transversality by means of gauge-transformations. In the case of simple poles, this trivialization can be naturally continued to the poles with tangency, since there are gauge transformations conjugating $\mathcal{F}$ to one of the standard forms not depending on the parameter. In the case of multiple poles this is not true. In order to trivialize the foliation, we would a priori need to reduce the remaining set of parameters to a submanifold of codimension $l-1$ for each pole of multiplicity l in which a tangency occurs. Let us consider an example illustrating this situation in a germified neighborhood of a double pole.

Example 5.8. *Consider the foliation on* $\mathbf{C}\times\mathbf{P}^1$ *given by*

$$\mathrm{d}y - y\theta\frac{\mathrm{d}x}{x},$$

where θ *is a non-zero complex number. We construct an isomonodromic deformation by an elementary transformation in* $(0,t)$ *with parameter* $t\in T$ *where* $(T,t_0)=(\mathbf{C},\epsilon)$ *such that the section* $\Sigma=\{y=1\}$ *is sent to infinity:*

$$\hat{y}=\frac{1}{x}\frac{y-t}{y-1}$$

For each parameter t *we obtain a Riccati foliation with singularities of second order. We normalize such that the coefficient of* $\tilde{y}^2$ *is* 1*:*

$$\tilde{y}=\frac{\theta}{1-t}\hat{y}.$$

We get the following global foliation on $\mathbf{C}\times T\times\mathbf{P}^1$*:*

$$\mathrm{d}\tilde{y}+\tilde{y}^2\mathrm{d}x+\tilde{y}\left(1-\frac{\theta(1+t)}{(1-t)}\right)\frac{1}{x}\mathrm{d}x+\frac{\theta}{(1-t)^2}\left(\frac{\theta t}{x^2}\mathrm{d}x-\frac{1}{x}\mathrm{d}t\right).$$

The restriction of this isomonodromic deformation to the section $\{\tilde{y}=\infty\}$ *is given by*

$$x^2\mathrm{d}x=0,$$

which does not depend on the parameter t*.*
After any holomorphic gauge-coordinate-transformation of type $(t,\varphi(x),\phi(t,x)\cdot\tilde{y})$*, the 1-form defining* $\mathcal{F}$ *will still have a non-trivial* $\mathrm{d}t$*-part, since the* $\mathrm{d}t$*-part has a pole which cannot disappear.*

This example shows that if we trivialize first the base curve of the universal isomonodromic deformation $\mathcal{F}$ in restriction to T', then it might later be impossible to trivialize the foliation if we want to keep the base curve trivial, unless we use some further restriction of the parameter space. But since we want the parameter space T' to satisfy (24), we have to refine our argumentation.

c) **Tangencies in singular points of the foliation**
Consider now a neighborhood of a tangency persisting in a singularity of the foliation. Consider a chart $T_k\times U\times\mathbf{P}^1$ containing such a singularity, with coordinates (t,x,y), such that $\mathcal{F}$ is constant in t. In particular, the position of the singularities and thus the position of the tangency shall not depend on t in these coordinates. Up to a gauge-coordinate-transformation constant in t we may suppose the tangency locus to be given by $\{x=0,y=0\}$. Then $\mathcal{F}$ and Σ are of the form

$$\mathcal{F}:\ \mathrm{d}y+\frac{\alpha(x)}{x^l}y^2+\frac{\beta(x)}{x^l}y+x\frac{\gamma(x)}{x^l},\quad \Sigma:\ \{y=xf(x,t)\},\tag{30}$$

where α,β,γ are holomorphic 1-forms, $l\geq 1$ and f is a holomorphic function. If we now apply an elementary transformation with center $\{x=0,y=0\}$, the resulting foliation will still be constant with respect to t. In fact the parameter does not interfere at all. The elementary transformation $\widehat{y} = \frac{y}{x}$ provides a new foliation $\widehat{\mathcal{F}}$ and a new section $\widehat{\Sigma}$:

$$\widehat{\mathcal{F}} :\ \mathrm{d}\widehat{y} + \frac{\alpha(x)}{x^{l-1}}\widehat{y}^2 + \left(\frac{\beta(x)}{x^l} + \frac{1}{x}\mathrm{d}x\right)\widehat{y} + \frac{\gamma(x)}{x^l}, \quad \widehat{\Sigma} :\ \{\widehat{y} = f(x,t)\}$$

and $s :\ \{x = 0, \widehat{y} = \infty\}$ is the center of the inverse elementary transformation. Note that the reduced induced foliation on $\widehat{\mathcal{F}}|_{\widehat{\Sigma}}$ did not change with respect to $\mathcal{F}|_{\Sigma}$.

Lemma 5.9. *The number of tangencies between $\widehat{\mathcal{F}}$ and $\widehat{\Sigma}$ is strictly smaller than the number of tangencies between $\mathcal{F}$ and Σ. The order of the pole at $\{x = 0\}$ will either stay the same or decrease as well.*

Proof: To prove this result, it is more convenient to consider other coordinates (t, x, y), where Σ is given by $\{y = 0\}$ and the tangency is still given by $\{x = 0, y = 0\}$. Let ν be the order of the tangency and let l be the order of the pole in $\{x = 0\}$. Recall that we are supposing $\nu, l > 0$. Since the foliation $\mathcal{F}$ is locally trivial up to gauge-coordinate-transformations, the defining 1-form ω satisfies $(\mathrm{d}\omega)_\infty \leq (\omega)_\infty$. The holomorphic 1-form $\widetilde{\omega} = x^l\omega$ then is of the form

$$\widetilde{\omega} :\ x^l\mathrm{d}y + y^2(a_0\mathrm{d}x + xa_1\mathrm{d}t) + y(b_0\mathrm{d}x + xb_1\mathrm{d}t) + x^\nu c_0\mathrm{d}x + xc_1\mathrm{d}t,$$

where $a_0, a_1, \ldots, c_1$ are holomorphic functions in (t, x) and $c_0(0, t) \neq 0$. In these coordinates, the elementary transformation centered in $(0, 0)$ is given by $\widehat{y} = \frac{y}{x}$. By this transformation we get

$$\widehat{\widetilde{\omega}} :\ x^l\mathrm{d}\widehat{y} + x\widehat{y}^2(a_0\mathrm{d}x + xa_1\mathrm{d}t) + \widehat{y}((b_0 + x^{l-1})\mathrm{d}x + xb_1\mathrm{d}t) + (x^{\nu-1}c_0\mathrm{d}x + c_1\mathrm{d}t)$$

and a new section $\widehat{\Sigma} :\ \{\widehat{y} = 0\}$ corresponding to Σ. If the 1-form $\widehat{\widetilde{\omega}}$ is reduced, then $\widehat{\mathcal{F}}$ has a pole of order $l \geq 1$ at $\{x = 0\}$ and the order of tangency with the section $\{\widehat{y} = \infty\}$ is $\nu - 1$. Otherwise the order of the pole will decrease and the order of tangency, too. $\square$

We can iterate this procedure for every tangency located at a singularity until the foliation becomes either transverse to the section, or non-singular in restriction to the section. Denote successively by $\mathrm{elm}_1, \ldots, \mathrm{elm}_\eta$ the necessary elementary transformations. Let us denote by $\widehat{\mathcal{F}}$ the foliation and by $\widehat{\Sigma}$ the section obtained after these η elementary transformations. Note that for every parameter $t \in T_k$, we have

$$\#\mathrm{tang}(\widehat{\mathcal{F}}_t, \widehat{\sigma}_t) \leq \#\mathrm{tang}(\mathcal{F}_t, \sigma_t) - \eta.$$

Now we may apply the argumentation of paragraph a) to $(\widehat{\mathcal{F}}, \widehat{\Sigma})$. We thus can trivialize the curve $\mathcal{X}$ and the position of the poles as well as the foliation $\widehat{\mathcal{F}}$ by restricting the parameter space T_k to a subspace $\widehat{T}'$ of codimension $\#\mathrm{tang}(\widehat{\mathcal{F}}_{t_k}, \widehat{\sigma}_{t_k})$. In particular, this means there is a gauge-coordinate transformation

$$(t, \widetilde{x}, \widetilde{\widehat{y}}) = (t, \varphi(t, x), \phi(t, x) \cdot \widehat{y})$$

with $\varphi(t,0)=0$ on $\widehat{T}'\times U\times\mathbf{P}^1$, where $\widehat{T}'$ is a submanifold of T_k, such that $\widehat{\mathcal{F}}$, $\widehat{\Sigma}$ and s are given in the new coordinates by

$$\begin{array}{llll} \widehat{\mathcal{F}}: & \mathrm{d}\widetilde{\widetilde{y}}+\frac{1}{\widetilde{x}^l}\widetilde{\widetilde{y}}^2\mathrm{d}\widetilde{x}+\frac{c(\widetilde{x})}{\widetilde{x}^l}\mathrm{d}\widetilde{x}, & \widehat{\Sigma}: \ \{\widetilde{\widetilde{y}}=\infty\}, & s: \ \{\widetilde{x}=0,\widetilde{\widetilde{y}}=\phi(t,0)\cdot\infty\}, \\ \text{resp.} & & & \\ \widehat{\mathcal{F}}: & \mathrm{d}\widetilde{\widetilde{y}}, & \widehat{\Sigma}: \ \{\widetilde{\widetilde{y}}=\infty\}, & s: \ \{\widetilde{x}=0,\widetilde{\widetilde{y}}=\phi(t,0)\cdot\infty\}, \end{array}$$

if $\widehat{\mathcal{F}}$ has a singularity of order $\hat{l}\geq 1$ at $\{x=0\}$, respectively if $\widehat{\mathcal{F}}$ is non-singular at $\{x=0\}$. Now we can choose a subvariety T' of codimension 1 in $\widehat{T}'$ such that $\phi(t,0)\cdot\infty\equiv\iota$ is constant for $t\in T'$. For a generic initial parameter, the germ (T',t_k) is a submanifold. In restriction to this new parameter space T', we now can apply the inverse elementary transformation $\widetilde{y}=\frac{\widetilde{\widetilde{y}}-\iota}{\widetilde{x}}$ with center in s. Note that again, this means applying a constant elementary transformation to a constant foliation and a constant section. We obtain a constant foliation $\widetilde{\mathcal{F}}$ and a constant section $\widetilde{\Sigma}$, given in these coordinates respectively by

$$\begin{array}{lll} \widetilde{\mathcal{F}}: & \mathrm{d}\widetilde{y}+\frac{1}{\widetilde{x}^{\hat{l}-1}}\widetilde{y}^2\mathrm{d}\widetilde{x}+\left(\frac{2\iota}{\widetilde{x}^{\hat{l}}}+\frac{1}{\widetilde{x}}\right)\widetilde{y}\mathrm{d}\widetilde{x}+\left(\frac{\iota^2+c(\widetilde{x})}{\widetilde{x}^{\hat{l}+1}}\right)\mathrm{d}\widetilde{x}, & \widetilde{\Sigma}: \ \{\widetilde{y}=\infty\}, \\ \text{and} & & \\ \widetilde{\mathcal{F}}: & \mathrm{d}\widetilde{y}+\frac{1}{\widetilde{x}}\widetilde{y}\mathrm{d}\widetilde{x}, & \widetilde{\Sigma}: \ \{\widetilde{y}=\infty\}. \end{array}$$

In proceeding in that way successively with all the inverse elementary transformations, beginning from elm_η^{-1} to elm_1^{-1}, we obtain a submanifold T' of codimension η in $\widehat{T}'$, such that all the centers of the elementary transformations are constant in t. Denote by $(\widetilde{\mathcal{F}},\widetilde{\Sigma})$ the foliation and the section we obtain after applying the inverse elementary transformations. By construction, $(\widetilde{\mathcal{F}},\widetilde{\Sigma})$ can be obtained from the reduced foliation $\mathcal{F}|_{T'}$ and the section $\Sigma|_{T'}$ in restriction to T', by holomorphic gauge-coordinate transformations. In these coordinates, the isomonodromic deformation with parameter space T' is trivial. Finally, $\eta+\#\mathrm{tang}(\widehat{\mathcal{F}}_{t_k},\widehat{\sigma}_{t_k})\leq\#\mathrm{tang}(\mathcal{F}_{t_k},\sigma_{t_k})$ codimensions are sufficient to locally trivialize the curve, the position of the poles and the foliation. Again, this implies thet the transition maps (φ,ϕ) are also constant in t and we have

$$((\mathcal{P}\to\mathcal{X},\mathcal{F}),\mathcal{D})|_{T'}\cong((P_{t_k}\to X_{t_k},\mathcal{F}_{t_k}),D_{t_k})\times T',$$

where T' satisfies (24).

This concludes the proof of theorem 1.1

6 Further comments

In this section, we shall discuss the necessity of the hypotheses of theorem 1.1.

6.1 Result for reducible connections

The irreducibility of the initial connection ∇_0 is needed in the above proof only to ensure that the section σ_0 of P_0 having minimal self-intersection number is not invariant by the

foliation $\mathbf{P}(\nabla_0)$.
Suppose now that (E_0, ∇_0) is a reducible connection. Let s_0 be a section invariant by the associated Riccati foliation. Then s_0 defines a global section $\mathcal{S} = (s_t)_{i\in T}$ invariant by the Riccati foliation associated to the universal isomonodromic deformation $(E_t, \nabla_t)_{t\in T}$. This section defines a holomorphic family of line bundles $(L_t)_{t\in T}$ provided with an integrable connection $(\zeta_t)_{t\in T}$ induced by ∇. Since the Euler class may not vary along such a family, for each parameter $t \in T$ we have

$$\text{deg}L_t = \text{deg}L_0.$$

From

$$s_t \cdot s_t = \deg\det(E_t) - 2\text{deg}L_t = -2\text{deg}L_t$$

we may conclude that the self-intersection number of the invariant section s_t is constant along the isomonodromic deformation. The minimal self-intersection number thus may never be greater than $s_0 \cdot s_0$ along the universal isomonodromic deformation. If the initial connection has two distinct invariant sections, then it is in fact decomposable into a direct sum of two rank 1 connections. Along the universal isomonodromic deformation, the underlying vector bundle then remains decomposable; its degree of stability is constant and equal to

$$\tilde{k}_0 = \min\{s_0 \cdot s_0 \mid s_0 \text{ section of } E_0 \text{ invariant by } \nabla_0\}. \tag{31}$$

If the initial connection is reducible but not decomposable, there is exactly one invariant section s_0 of the initial connection. We then define $\tilde{k}_0 = s_0 \cdot s_0$, conformal to (31). Let k be an integer strictly smaller than $\tilde{k}_0$. Then for each parameter $t \in T_k$, where T_k is defined as before, any section σ_t of P_t with self-intersection number k is not invariant by the foliation $\mathcal{F}_t$. We then may apply the proof of theorem 1.1 to show $\text{codim}(T_k) \geq g - 1 - k$. It follows the following, sharper version of theorem 1.1.

Theorem 6.1. *Let $(E_t, \nabla_t)_{t\in T}$ be the universal isomonodromic deformation of a tracefree meromorphic rank 2 connection over some Riemann surface of genus g. We define $k_0 = g$ if the initial connection is irreducible, and $k_0 = \min(g, \tilde{k}_0)$ with $\tilde{k}_0$ as in (31) else. Then*

$$\begin{cases} T_{k_0} = T \\ \text{codim}(T_k) \geq g - 1 - k \quad \forall k < k_0. \end{cases}$$

In particular, the vector bundle underlying the universal isomonodromic deformation is generically maximally stable if, and only if, the initial connection is irreducible or $\tilde{k}_0 \geq g$.

Let us now construct examples of reducible tracefree rank 2 connections $(E_0 \to X_0, \nabla_0)$ such that the vector bundle underlying their universal isomonodromic deformation $(E_t \to X_t, \nabla_t)_{t\in T}$ is not generically maximally stable. In other words, let us construct examples of tracefree rank 2 connections $(E_0 \to X_0, \nabla_0)$ over Riemann surfaces of genus g, which have invariant sublinebundles L_0 such that

$$-2\text{deg}L_0 < g - 1.$$

Example 6.2. *Let $\rho : \pi_1(X_0) \to \text{Aff}(\mathbf{C})$ be a representation, where we identify*

$$\text{Aff}(\mathbf{C}) \cong \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} \;\middle|\; \lambda \in \mathbf{C}^*, \mu \in \mathbf{C} \right\}.$$

Then the Riemann-Hilbert correspondence provides a non-singular rank 2 connection over X_0 with monodroy ρ, which will have an invariant sublinebundle L_0. In particular, the line bundle L_0 can be provided with a non-singular connection. Thus $\deg L_0 = 0$ *(see [Wei38]). This provides such an example if $g \geq 2$.*

Example 6.3. *Let (E_0, ∇_0) be an irreducible tracefree rank 2 connection. Denote by σ_0 an invariant section. By an even number of elementary transformations with center on σ_0 and renormalization afterwards (to restore the tracefreeness-condition), we can decrease arbitrarily the self-intersection number of σ_0.*

6.2 Remarks on undeformable connections

Consider tracefree rank 2 connections on a Riemann surface of genus g with n poles, counted with multiplicity. Assume $3g - 3 + n \leq 0$. We would like to know if they are defined on maximally stable bundles. Note that these connections are undeformable in the genus 0 case.

Proposition 6.4. *If such a connection is irreducible, then it is defined on a maximally stable bundle.*

Proof: Let σ be a section of the associated $\mathbf{P}^1$-bundle with minimal self-intersection number. If the connection is irreducible, we know that σ is not invariant by the connection. Then the formulas of M. Brunella and M. Nagata imply

$$\begin{cases} 2 - n \leq \sigma \cdot \sigma \leq 0 & \text{for } g = 0 \\ \quad 0 \leq \sigma \cdot \sigma \leq 1 & \text{for } (g, n) = (1, 0). \end{cases}$$

We notice that for $g = 0$, the assumption $n \in \{0, 1\}$ leads to a contradiction. On the other hand, the vector bundle is clearly maximally stable in the remaining cases $(g, n) = (0, 2), (0, 3)$ and $(1, 0)$. □

According to the Riemann-Hilbert correspondence, a logarithmic rank 2 connection is reducible if, and only if, the corresponding monodromy representation is reducible. Since the fundamental groups of elliptic curves and the fundamental group of the Riemann sphere with at most 2 punctures is abelian, for each choice of residues, the corresponding logarithmic rank 2 connections are reducible. Yet a generic rank 2 connection with three poles (counted with multiplicity) on the Riemann sphere is irreducible. It defines a hypergeometric (Gauß or confluent) differential equation (see [Inc44]).

6.3 Remarks on connections with trace

In the spirit of [Mal83a], [Mal83b],[Pal99] and [Kri02], one can also construct universal isomonodromic deformations with varying trace, provided that the leading terms of the connection matrices are non-resonant. Consider a meromorphic rank 2 connection (E_0, ∇_0) on a Riemann surface X_0 with arbitrary trace. If the degree of the line bundle $\det(E)$ is even, then there is a tracefree meromorphic rank 2 connection $(\widetilde{E}_0, \widetilde{\nabla}_0)$ and a meromorphic rank 1 connection (L_0, ζ_0) on X_0, such that

$$(E_0, \nabla_0) = (L_0, \zeta_0) \otimes (\widetilde{E}_0, \widetilde{\nabla}_0).$$

Let us suppose that both ζ_0 and $\widetilde{\nabla}_0$ have the same polar divisor (counted with multiplicity) as ∇_0. As before, the parameter space of the universal isomonodromic deformation of (E_0, ∇_0) will have a natural product structure. One factor $\mathcal{T}$ comes from the deformation of the punctured curve and will have dimension $3g-3+m$. The deformation by local jets of the tracefree connection $(\widetilde{E}_0, \widetilde{\nabla}_0)$, provides a second factor J of the parameter space, with dimension $n-m$. As for the deformation by local jets of the rank 1 connection (L_0, ζ_0) over the universal curve, we get an additional factor J' of the parameter space, with dimension $n-m$. Finally, we would obtain a universal isomonodromic deformation $(\mathcal{E}, \nabla)$ with parameter space T of dimension $3g-3+2n-m$. An analog construction has been done in the paper of John Palmer [Pal99] for the genus 0 case.

Suppose now that the initial connection is irreducible. We then can apply the proof of theorem 1.1 to the projective deformation $(\mathbf{P}(\mathcal{E}), \mathbf{P}(\nabla))$, using the fact that the degree of stability of a rank 2 vector bundle only depends on the associated projective bundle and thus is invariant by deformation along J'. We can thus trivialize $(\mathbf{P}(\mathcal{E}), \mathbf{P}(\nabla))$ in restriction to a subspace of codimension $2g-2+n-k$ in T_k, where T_k is defined as in section 1.1. On this new parameter space, the trace connection of $(\mathcal{E}, \nabla)$ may still vary non-trivially. However, by further restriction of the parameter space by $\dim(J') = n-m$ codimensions, we will be able to trivialize the deformation of the trace connection over the already trivialized curve. We obtain again

$$\operatorname{codim}(T_k) \geq g-1-k.$$

In particular, the vector bundle underlying such a universal isomonodromic deformation will still be generically maximally stable.

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